CLOSED SUBALGEBRAS OF $L^1(T)$

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1. Introduction. If T is the circle group then $L^1(T)$ is a Banach algebra where the multiplication is given by convolution. $||f||$ and $\hat{f}$ will denote the norm and Fourier transform of a function in $L^1(T)$. If A is a closed subalgebra of $L^1(T)$ then A determines an equivalence relation, R_A , on the integers by $n \sim m$ if $\hat{f}(n) = \hat{f}(m)$ for all $f \in A$. R_A has one possibly infinite equivalence class, E_0 , consisting of all n such that $\hat{f}(n) = 0$ for all $f \in A$. The other classes, denoted by E_i , are finite.

Conversely if R is an equivalence relation on the integers with at most one infinite equivalence class, there is a closed subalgebra A such that $R_A = R$. There are two obvious candidates for this: B^R, consisting of the $f \in L^1(T)$ such that $\hat{f}$ is constant on the classes of R and 0 on E_0 ; and B_R , the closed subalgebra generated by the polynomials I_i such that $\hat{I}_i$ is the characteristic function of E_i ($j \neq 0$).

Rudin, [6; 231], has shown that if A is a closed subalgebra with $R_A = R$, then

$$B_R \subset A \subset B^R.$$

Kahane, [2] and [3], has constructed an example where $B_R \neq B^R$. This is equivalent to saying that the algebra B^R is not generated by its idempotents; i.e., there is $f \in B^R$ such that the Fourier series for f can be written

$$f(t) \sim \sum_i a_i I_i(t),$$

but f cannot be approximated in $L^1(T)$ by polynomials of the form

$$g(t) = \sum_i b_i I_i(t).$$

It can be shown that the particular functions, as above, which Kahane constructs are not in $L^p(T)$ if $p > 1$. In this paper we shall construct an example as follows:

Example A. *There is an equivalence relation R on the integers and a function f such that*:

(a) $f \in \bigcap_{r<2} L^r(T)$

(b) $f \in B^R$ *but* $f \notin B_R$.

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