SPHERICAL MODIFICATIONS AND COVERINGS BY CELLS

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As is well known, the minimum possible number of critical values of any Morse function on a differentiable manifold and the minimum number of cells needed to cover a differentiable manifold are not spherical modification invariants. It is the purpose of this paper to establish the existence of a class of n-manifolds for which these two numbers are modification invariants. Lower bounds to the number of such manifolds (modulo cobordism) will be given for even dimensional manifolds, (Corollary 6.2). It will also be shown that these numbers can be invariant only when they equal $n + 1$.

Unless stated otherwise, an n-manifold M is an n-dimensional, compact, connected, smooth manifold without boundary. A cell of M is a subset of M homeomorphic to an n-cell. The cell number $C(M)$ of M is the minimum number of cells needed to cover M. It may be assumed the M is covered by the interiors of the cells. The strong category of M, Cat M, is the minimum number of contractable open sets needed to cover M.

Note that $C(M)$ is not necessarily equal to the minimum number of coordinate neighborhoods needed to cover M. For example, T^2(2-torus) can be covered with two coordinate neighborhoods but $C(T^2) = 3$.

Let M be an n-manifold and $f: M \to R$ (Real numbers) a smooth function. A point $p \in M$ is a critical point of f if for some coordinate system $(x_1, \cdots, x_n)$ about p,

$$\left.\frac{\partial f}{\partial x_1}\right|_p = \cdots = \left.\frac{\partial f}{\partial x_n}\right|_p = 0$$

and it is a non-degenerate critical point with index λ if for some coordinate system $(x_1, \cdots, x_n)$, $f(x_1, \cdots, x_n) = \text{constant} - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ for some integer λ, $0 \leq \lambda \leq n$ and $p = (0, \cdots, 0)$. A smooth function $f: M \to R$ is a Morse function if all of its critical points are non-degenerate. The image of a critical point is a critical value. Clearly for a Morse function the critical points are isolated and thus by compactness are finite in number, hence the number of critical values is also finite. Let $\mu(M)$ be the minimum, over all Morse functions f on M, of the number of critical values of f.

Theorem 1. *If M is an n-manifold, then* Cat $M \leq C(M) \leq \mu(M) \leq n + 1$.

Proof. That Cat $M \leq C(M)$ is trivial. It is known that there exists a Morse function on any n-manifold with $\leq n + 1$ critical values [7; 44]; hence $\mu(M) \leq n + 1$. It remains to show that $C(M) \leq \mu(M)$. This result is essentially obtained in [4], however, an alternate proof will be indicated here.

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