ARTINIAN RINGS WITH A CYCLIC QUASI-REGULAR GROUP

By Irwin Fischer and Klaus E. Eldridge

In a recent paper [2] the authors have found all Artinian rings with a cyclic group of units. Here, we extend these results to rings without a unit element. The group of units of the ring is replaced by the group of its quasi-regular elements. Of course, these two groups are isomorphic if the ring has a unit element.

As before, it turns out that these rings must all be finite. Non-commutative rings occur only in the case of characteristic two.

We continue to use the notation and terminology introduced in [2]. In addition: the number of elements in R will be denoted by $|R|$; the group of quasi-regular of R will be designated by $^{\circ}R$; and the additive group of R will be denoted by ^{+}R.

1. Rings with an abelian, finitely generated quasi-regular group. Our first concern is to prove that the rings we are interested in are actually finite. In this connection we have the following result.

Theorem 1. *Let R be an Artinian ring. If the quasi-regular group of R is abelian and finitely generated, then R is finite.*

Proof. Let J be the radical of R, and let m be its index of nilpotence. We start by showing that J is finite.

Let $\{u_1, u_2, \cdots, u_n\}$ be a set of generators of $^{\circ}J$, the quasi-regular group of J. Assume that $m = 2$. Then $J^2 = (0)$, and so $\{u_1, u_2, \cdots, u_n\}$ also form a set of generators of the additive group of J. Hence, by a result of Szasz [4, Theorem 7], ^{+}J is isomorphic to the direct sum of a finite number of finite cyclic groups, and hence J is finite in this case.

Suppose now that $m > 2$. Then consider $R' = R/J^2$, and let J' be the radical of R'. By the corollary to Lemma 1 of [2], R' satisfies the hypothesis of the theorem. Since J' is nilpotent of index 2, we conclude by the case $m = 2$ above that J' is finite.

Let q be the characteristic of J', considered as a ring. From $qJ' = (0)$ we have $qJ \subset J^2$. We then see that $q^{m-1}J \subset J^m = (0)$, hence J is of finite characteristic. Let r be this characteristic.

Denote the quasi-inverse of u_i by v_i, $1 \leq i \leq n$. Then every element in $^{\circ}J$ is a polynomial in $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$, of total degree less than m, with coefficients in Z_r, the ring of integers modulo r. Hence J is a finite Z_r − module, and so is finite.

Now if $R = J$, we are done. Otherwise, we again apply the corollary to

Received July 20, 1967. This research was in part supported by the National Science Foundation.