SIDE APPROXIMATIONS IN CRUMPLED CUBES

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1. Introduction. Bing [2] has proved in what is known as 'The Side Approximation Theorem' that any topological 2-sphere in E^3 can be almost approximated from its interior. In fact, for each such 2-sphere S there is a polyhedral 2-sphere $h(S)$ homeomorphically close to it and inside it except for a finite number of mutually exclusive small disks on $h(S)$ which intersect S in a finite number of mutually exclusive small disks. In Bing's proof the small disks on $h(S)$ may intersect the exterior of S as well as S. One might ask under what conditions is it possible to side approximate S from the interior of S missing the exterior of S (see Property P below). In §4 it is shown that a sphere tame from its interior modulo a zero-dimensional set can be side approximated from its interior missing the exterior. In §4 it is shown that the sphere described by Gillman [8] cannot be side approximated from its interior missing its exterior.

Burgess [7], Hempel [9], [10] and Loveland [13] have given partial answers to the following question raised by Bing [5]. Under what conditions is there a mapping, arbitrarily close to the identity, of a 2-sphere S into the interior of S; i.e., under what conditions is a 2-sphere free from its interior? In particular, must S be tame from the interior of S if S is free from the interior of S? In §4 another partial answer is given: If a 2-sphere is locally free from its interior and can be side-approximated from its interior missing its exterior, then it is tame from its interior. Also, it is shown that if a 2-sphere can be side-approximated from its interior missing its exterior, and its interior is an open 3-cell, then it is tame from its interior modulo a single point.

2. Definitions and notation. The usual set theoretic symbols for union, intersection, membership and inclusion are used. If x and y are points, then $\rho(x, y)$ denotes the distance between x and y. If A and B are sets then $\rho(A, B)$ is the distance between A and B. The diameter of a set A is denoted by Diam A. An ϵ-neighborhood of a set A is denoted by $N(A, \epsilon)$. The exterior, interior, and boundary of A are denoted respectively by Ext A, Int A, and Bd A, and it will be clear from context whether these are meant in the set theoretic, combinatorial, or complementary domain sense. An arc with end points p and q is denoted by pq. The closure of set A is denoted by Cl A. Euclidean n-space is denoted by E^n.

A set A is an *ϵ-set* if Diam $A < \epsilon$. A map f from set A is an *ϵ-map* if $\rho(f(x), x) < \epsilon$ for each $x \in A$. A set M is *free from set U* if for each $\epsilon > 0$ there is an ϵ-map

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