RINGS WITH NOETHERIAN SPECTRUM

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0. Introduction. Let R and R' denote commutative rings with identity. A radical ideal of R is an intersection of prime ideals, and a J-radical ideal is an intersection of maximal ideals. If R has ascending chain condition (a.c.c.) for radical ideals, then $R[X]$ does also; while if R has a.c.c. for J-radical ideals, it is no longer true that $R[X]$ does (§2). However, our main theorem asserts that if R' is a finite integral extension of R and R has a.c.c. for J-radical ideals, then R' inherits this property (Theorem 3.6).

In topological terms (see [5-a, Chapter 0, §2; 5-b, Chapter 0, §14]), we are interested in the following properties of a space X, where X is either the prime or maximal spectrum of R: (1) X is noetherian, (2) every closed subset of X has finitely many irreducible components, and (3) X has finite combinatorial dimension. We investigate in particular which of these properties carry over to $R[X]$ or to a finite integral extension of R. We conclude by applying the main theorem to obtain a new proof of a result of Bass on the stable range of a finite commutative R-algebra.

Throughout the paper, the word "ring" stands for a commutative ring with identity; by "ideal" we mean an ideal different from the ring itself. We use $\subseteq$ and $\subset$ for weak and strong inclusion, respectively, while $A\backslash B = \{a \in A \mid a \notin B\}$. If R is a ring and $c \in R$, $c \neq 0$, then R_c denotes the ring of quotients of R with respect to the multiplicative system $\{c^n \mid n \geq 0\}$.

1. Definitions and basic consequences. Let R be a ring. If A is an ideal of R, then the *J-radical of A*, $J(A)$, is the intersection of all maximal ideals containing A. A is a *J-radical ideal* if $A = J(A)$. The *J-components of A* are the minimal members of the family of J-radical prime ideals containing A. Now let Y denote the maximal spectrum of R, i.e., Y is the set of maximal ideals of R with the Zariski topology: the closed sets of Y are the sets $V(A) = \{M \in Y \mid A \subseteq M\}$. In the usual way, ([5-a, Chapter 1, §1.1], or [2-a, Chapter 2, §4]), J-radical ideals correspond to closed subsets of Y, with J-radical prime ideals corresponding to closed irreducible subsets of Y. Thus the J-components of an ideal A correspond to the irreducible components of $V(A)$.

We shall study the following properties of R:

(N): The prime spectrum of R is noetherian, i.e., R satisfies the ascending chain condition for radical ideals.

(JN): The maximal spectrum of R is noetherian, i.e., R satisfies the ascending chain condition for J-radical ideals.

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