ORTHOGONAL POLYNOMIALS WITH BRENKE TYPE GENERATING FUNCTIONS

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Introduction. The polynomial generating functions of the form

$$A(w)B(xw) = \sum_{n=0}^{\infty} P_n(x)w^n \tag{1.1}$$

where

$$A(w) = \sum_{k=0}^{\infty} a_k w^k, \qquad a_0 \neq 0,$$

$$B(w) = \sum_{k=0}^{\infty} b_k w^k, \qquad b_n \neq 0 \quad (n \geq 0),$$

are particularly simple special cases of the generating functions of "generalized Appell type" studied by Boas and Buck [2]. They were apparently first studied by Brenke [3] who noted that the Hermite and Laguerre polynomials were among those sets generated by (1.1). Brenke also asked whether there are other sets of orthogonal polynomials with generating functions of this form.

An affirmative answer was provided in the work of Geronimus [8] who considered orthogonal polynomials of the form

$$P_n(x) = \sum_{k=0}^{n} a_{n-k} b_k \pi_k(x), \qquad n = 0, 1, 2, \cdots \tag{1.2}$$

where

$$\pi_0(x) = 1, \qquad \pi_k(x) = \prod_{i=1}^{k} (x - x_i) \qquad (k \geq 1).$$

Geronimus obtained a set of necessary and sufficient conditions on $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ for $\{P_n(x)\}$ to be an orthogonal set and exhibited a number of special cases explicitly. He did not determine all such cases explicitly however.

It is obvious that the "Brenke polynomials" generated by (1.1) are the special cases, $x_i = 0$, of (1.2). In this paper, we wish to determine explicitly all orthogonal polynomial sets generated by (1.1). In a sense, this will complement the work of Meixner [11] who determined all orthogonal polynomials generated by

$$A(w) \exp [xG(w)] = \sum_{n=0}^{\infty} P_n(x)w^n,$$

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