THREAD ACTIONS

By David Stadtlander

Introduction. If T is a topological semigroup and X is a Hausdorff space, an Act (act) is a continuous function μ from $T \times X$ onto (into) X such that $\mu[s, \mu(t, x)] = \mu(st, x)$ for each $s, t \in T$ and $x \in X$. We say that T Acts (acts) on X. If $S \subset T$ and $A \subset X$, we write SA rather than $\mu(S, A)$ when no confusion is likely.

Our topological semigroup notation will follow that found in [16]. We shall use $E(T)$ to denote the set of idempotents in the semigroup T and $K(T)$ to denote the minimal ideal of T. $\mathcal{L}$ and $\mathcal{H}$ are the equivalence relations introduced by Green [6]. They are defined as follows:

$$t\mathcal{L}s \quad \text{iff} \quad t \cup Tt = s \cup Ts$$

$$t\mathcal{H}s \quad \text{iff} \quad t \cup Tt = s \cup Ts \quad \text{and} \quad t \cup tT = s \cup sT.$$

A semigroup T is normal if $Tt = tT$ for each $t \in T$. In a normal semigroup the Green relations coincide and are congruences. In a compact topological semigroup T, $K(T)$ and $E(T)$ exist as non-void closed sets while for $e \in E(T)$, the $\mathcal{H}$-class containing e (denoted by $H(e)$), is a maximal subgroup of T and is a compact topological group.

If $\leq$ denotes a quasi-order on the space X, for $A \subset X$ we set $L(A) = \{y \in X \mid y \leq x \text{ for some } x \in A\}$. We say $\leq$ is continuous if $L(A^*) \subset L(A)^*$ for each $A \subset X$. We say $\leq$ is closed if $\{(x, y) \mid x \leq y\}$ is a closed subset of $X \times X$. We recall a result of W. Strother [15] which (for metric spaces) states:

(1) If $\leq$ is a quasi-order on X, it is closed and continuous iff for any sequence $\{x_n\}$ in X with $x_n \to x$, we have $L(x_n) \to L(x)$.

Let the semigroup T Act on the space X. We consider now a method of partitioning the space X according to the mobility of its points under mappings from the semigroup T. In particular, we define a quasi-order $\leq$ on X by $x \leq y$ iff $x \in y \cup Ty$ and we refer to $\leq$ as the natural quasi-order on X (induced by the Action of T). We also consider the associated equivalence relation δ on X ($x\delta y$ iff $x \leq y$ and $y \leq x$) and let $\delta[x]$ denote the equivalence class containing x. The δ-classes are in some ways reminiscent of group orbits and the hyperspace, of a space Acted on by a thread. While in general this analogy is rough, it is not surprising that it is an excellent one for Actions by compact normal semigroups [14, §2]. One easily sees that for $A \subset X$, $L(A) = A \cup TA$. From the continuity of the Act function we then have $L(A^*) = A^* \cup TA^* \subset (A \cup TA)^* = L(A)^*$, so $\leq$ is continuous. A standard argument using Theorem

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