THE FRIEDRICHS EXTENSION OF CERTAIN SINGULAR DIFFERENTIAL OPERATORS

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Introduction. Let T be a symmetric operator in a Hilbert space H, with domain $D(T)$ dense in H. If the deficiency indices of T are equal, then T has self-adjoint extensions [1; §XII.4]. In the general theory, the class of all self-adjoint extensions of a given symmetric operator with equal finite deficiency indices is described in terms of boundary conditions. If the symmetric operator T is also semi-bounded, a particular ("ausgezeichnete") self-adjoint extension having the same bound has been constructed by Friedrichs [2], [1; §XII.5], and [4; §124]. This extension is an "extension by closure" since it is obtained by restricting the adjoint operator T^* to the closure of $D(T)$ in a "new" inner product. In [3], Friedrichs gave a boundary condition description of this extension in the case of the second-order Sturm–Liouville differential operator under certain restrictions on the coefficients. Rellich [5] later obtained the "ausgezeichnete" boundary conditions with less restrictions on the coefficient functions and in addition two new characterizations of the boundary conditions were obtained.

In this paper, we shall characterize the Friedrichs extension in terms of explicit boundary conditions for a certain class of higher order singular differential operators. It will be apparent that the technique used could be applied to a much wider class of operators. We have chosen to restrict ourselves to the special class considered here not only for convenience of exposition, but also because these specific operators appeared while the author was studying a convergence problem for the eigenvalues of certain Toeplitz matrices associated with Jacobi polynomials. The present results find application there.

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2. Statement of main results. Consider the differential operator

$$\tau u = -\frac{1}{m}[(pu')' - qu], \qquad 0 < x < 1 \tag{1}$$

where m, p, and q are all infinitely differentiable functions on (0, 1) and these functions satisfy

$$p(x) > 0, \quad m(x) > 0, \quad q(x) \geq 0, \quad \text{for} \quad x \in (0, 1). \tag{2}$$

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