SETS OF NEARLY TRIANGULAR MATRICES

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The object of this paper is to prove a number of theorems that give conditions under which matrices $A_1, \cdots, A_m$ with elements in a field K are simultaneously similar over K to a block triangular form of a certain type. As special cases of our results we recover known necessary and sufficient conditions in order that $A_1, \cdots, A_m$ be simultaneously similar over the algebraic closure of K to triangular matrices. This problem was first examined by McCoy [9], and subsequently by Eaves [3], [4], Goldhaber [6], Goldhaber and Whaples [7], Drazin, Dungey, and Gruenberg [2]. Later Drazin [1] extended the results of [2]. Our principal result is a sharpened version of the main McCoy-Drazin, Dungey, Gruenberg result [2, Theorem 1]. As well, we shall prove sharpened versions of the theorems of [1]. Recently Sinha [13] gave a sufficient condition involving multiplicative commutators to guarantee that nonsingular matrices $A_1, \cdots, A_m$ are simultaneously similar to triangular matrices. We present a short proof of Sinha's result, using the methods of this paper. Our final results will be a reexamination of the known fact (due to I. Schur) that any square matrix over the complex field is unitarily similar to a triangular matrix. As a special case of our Corollary 3 we obtain an improved real counterpart of Schur's result that does not seem to have been noticed previously and which contains as a special case a known diagonalization theorem for real normal matrices. Our proofs, using only well known and mostly elementary techniques in linear algebra, seem to yield our results more directly and easily than the proofs in [1], [2], [3], [4], [6], [7], [9], [13] (although it should be mentioned that some of our arguments are close to arguments in [9]). In particular, we are able to avoid most of the use of commutator identities (as in [1] or [13]) and we are also able to avoid much of the heavy use of proof by induction (as in [1], [2]). In some instances we have included proofs of known results, when these results follow naturally using our methods.

Throughout this paper K denotes an arbitrary field and L is the algebraic closure of K. All polynomials $p = p(x_1, \cdots, x_m)$ are to be in noncommutative variables $x_1, \cdots, x_m$. If matrix A is n-square, we say degree $A = n$. The

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