A NOTE ON DIMINISHING THE UNDECIDABLE REGION OF A RECURSIVELY ENUMERABLE SET

BY ROBERT A. DI PAOLA

1. Introduction. As in [2] and [3], by a "theory T" we mean a consistent re (recursively enumerable) extension of Peano arithmetic P; and by a "pair (T, α)" we mean such a theory T together with an RE-formula α which represents the axioms of T in Peano arithmetic P. In [2] and [3] the process of enlarging the pseudo-complement $\{N_\alpha(n)\}$ of an re set $S = \{n\}$ or, what is the same thing, of diminishing the undecidable region $\bar{S} - \{N_\alpha(n)\}$ of S with respect to the index n by adjunction to T of undecidable sentences of the form $x \not\varepsilon \{n\}$ was discussed (cf. also [1; 128]). Given any pair (T, α), it is an elementary fact that if $S = \{m\} = \{n\}$, the sets $\{N_\alpha(m)\}$ and $\{N_\alpha(n)\}$, and hence the undecidable regions $\bar{S} - \{N_\alpha(n)\}$ and $\bar{S} - \{N_\alpha(m)\}$, can differ greatly. What is more, for suitable m and n, with $\{m\} = S = \{n\}$, if $x \,\varepsilon\, (\bar{S} - \{N_\alpha(m)\}) \cap (\bar{S} - \{N_\alpha(n)\})$, the consequences of adding $x \not\varepsilon \{m\}$ or $x \not\varepsilon \{n\}$ to T as new axioms can be very different. In this note we limit examination of these divers consequences to two theorems, which do, nevertheless, show how disparate are the possibilities. We shall prove:

(1) If S is any re set and A is any re subset of $\bar{S}$, then there is an index n of S such that

- (a) for all $x \,\varepsilon\, \bar{S}$, $x \not\varepsilon \{n\}$ is undecidable;
- (b) for all $y \,\varepsilon\, \bar{S} - A$ and $x \,\varepsilon\, A$, $\vdash_T y \not\varepsilon \{n\} \rightarrow x \not\varepsilon \{n\}$ but not $\vdash_T x \not\varepsilon \{n\} \rightarrow y \not\varepsilon \{n\}$.

(2) If C is any creative set, then there is an index n of C such that whenever $x \,\varepsilon\, \bar{C} - \{N_\alpha(n)\}$ there can be found a y such that

- (a) $y \,\varepsilon\, \bar{C} - \{N_\alpha(n)\}$;
- (b) $\vdash_T x \,\varepsilon\, \{n\} \rightarrow y \not\varepsilon \{n\}$, and
- (c) not $\vdash_T y \not\varepsilon \{n\} \rightarrow x \not\varepsilon \{n\}$.

2. Background and notation. Our notation for notions of the theory of recursive functions is drawn from [1]. For the notation, basic definitions, and results concerning pseudo-complements we refer the reader to papers [2] and [3], of which we shall make frequent use. With respect to the function $\wedge$ of [3], we shall write $z_1 \wedge z_2 \wedge \cdots \wedge z_k$ for $((\cdots (z_1 \wedge z_2) \wedge z_3) \wedge z_4 \cdots) \wedge z_k$. Throughout the remainder, it is assumed that T is weakly ω-consistent.

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