ON THE w^*-SEQUENTIAL CLOSURE OF A BANACH SPACE IN ITS SECOND DUAL

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1. Let X be a real Banach space and J_X the canonical mapping from X into its second dual X^{**}. This paper is concerned with $K(X)$, the w^*-sequential closure of $J_X X$ in X^{**}. Conditions under which $K(X) = X^{**}$ are studied, and relationships involving $K(X)$, $K(Y)$, and $K(X/Y)$ are obtained, where Y is a subspace of X.

Let $\mathcal{S}$ be the class of all norm-closed subspaces of the dual X^* of X. If $S \in \mathcal{S}$ and $\{x_n\}_{n \in \omega}$ is a sequence in X, then $\{x_n\}$ will be said to be *S-Cauchy* if $\lim_n f(x_n)$ exists for each $f \in S$; thus a sequence is X^*-Cauchy if and only if it is w-Cauchy. Let $\mathcal{A}$ be the class of all $S \in \mathcal{S}$ such that for every bounded S-Cauchy sequence $\{x_n\}$ in X there exists a sequence $\{y_n\}$ in the subspace S_0 of X annihilated by S, such that $\{x_n + y_n\}$ is w-Cauchy. Let $\mathcal{B}$ be the class of all $S \in \mathcal{S}$ such that for every bounded S-Cauchy sequence $\{x_n\}$ in X there exist a sequence $\{w_n\}$ of averages far out in $\{x_n\}$ and a sequence $\{y_n\}$ in S_0 such that $\{w_n + y_n\}$ is w-Cauchy. Finally, let $\mathcal{C}$ be the class of all saturated $S \in \mathcal{S}$.

In §2 it is shown that (i) $K(X) = X^{**}$ if and only if $\mathcal{A} \subseteq \mathcal{C}$, and (ii) the condition $\mathcal{B} = \mathcal{S}$ is necessary but not sufficient in order that $K(X) = X^{**}$. If Y is a closed subspace of X, let β be the natural mapping from X onto the quotient space $Z = X/Y$ and let Y^0 be the annihilator of Y in X^*. In §3 it is shown that (iii) $\beta^{**}(K(X)) = K(Z)$ if and only if $Y^0 \in \mathcal{B}$; (iv) in particular, $Y^0 \in \mathcal{B}$ if Y^* is separable, and $Y^0 \in \mathcal{A}$ if there is a continuous projection from X onto Y; finally (v) if $Y^0 \in \mathcal{B}$, then $Y^{**}/K(Y)$ is isomorphic with $[Y^{00} + K(X)]/K(X)$, and $Z^{**}/K(Z)$ is isomorphic with $X^{**}/[Y^{00} + K(X)]$. These isomorphisms are analogous with isomorphisms obtained by Civin and Yood [2; 908]. Pertinent examples are given.

2. For every Banach space X it is obvious that $\mathcal{A} \subseteq \mathcal{B}$. It will be shown by examples that the inclusions $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{B}$, and $\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{A}$ can fail to be valid.

THEOREM 1. *If X is a Banach space, then $K(X) = X^{**}$ if and only if $\mathcal{A} \subseteq \mathcal{C}$.*

Proof. Let $K(X) = X^{**}$ and $S \in \mathcal{S} \setminus \mathcal{C}$. Then there exist an $f_0 \in (S_0)^0 \setminus S$ and, by the Hahn–Banach theorem, an $F \in X^{**}$ such that $F(f_0) \neq 0$, but $F(g) = 0$ for each $g \in S$. Since $K(X) = X^{**}$, there is a sequence $\{x_n\} \subset X$ such that $\lim_n f(x_n) = F(f)$ for each $f \in X^*$. Now $\{x_n\}$ is bounded [1; 123], and hence $\{(-1)^n x_n\}$ is bounded and has the property that $\lim_n g((-1)^n x_n) = 0$ for each

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