ANALYTIC EIGENVALUES AND EIGENVECTORS

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Introduction. With t as a real variable, consider the matrix

$$A(t) = \begin{bmatrix} 1 & t^2 \\ t^2 & 1 + t \end{bmatrix}.$$

The eigenvalues $\lambda_1(t)$, $\lambda_2(t)$ of $A(t)$ are the zeros of

$$f(t, \lambda) = \lambda^2 - (2 + t)\lambda + (1 + t - t^4).$$

Since $f(0, \lambda) = (\lambda - 1)^2$, $\lambda = 1$ is an eigenvalue of $A(0)$ of multiplicity two. Let us investigate the possibility of defining the eigenvalues of $A(t)$ as differentiable functions of t in a suitably small neighborhood of $t = 0$. If we appeal to the implicit function theorem concerning this question, we obtain no information for $\partial f/\partial\lambda = 0$ at $(0, 1)$. However, it is clear that

$$\lambda_1(t) = \tfrac{1}{2}[2 + t + t(1 + 4t^2)^{\frac{1}{2}}], \tag{1}$$

$$\lambda_2(t) = \tfrac{1}{2}[2 + t - t(1 + 4t^2)^{\frac{1}{2}}], \tag{2}$$

define eigenvalues of $A(t)$. Moreover, these eigenvalues are not only differentiable but *analytic functions* of t for $|t| < \frac{1}{2}$. (A function is said to be analytic in a neighborhood of t_0 if it can be represented as a convergent power series $\sum a_n(t - t_0)^n$ in this neighborhood.) Thus, at the origin where the implicit function theorem fails, we have $\dot\lambda_1(0) = 1$, $\dot\lambda_2(0) = 0$.

In the present example, the ability to define λ_1 and λ_2 as analytic functions of t is a consequence of the fact that for $|t| < \frac{1}{2}$ (and indeed for any t) the eigenvalues of $A(t)$ are real. In this paper we shall examine the subject of analytic eigenvalues under the assumption that the elements of the $n \times n$ matrix $A(t)$ are polynomials in t. As we shall see, if the eigenvalues of $A(t)$ are real for $a' < t < b'$, then they can be defined as analytic functions of t in any interval $a \leq t \leq b$ such that $a' < a < b < b'$.

By this we mean that there exist functions $\lambda_j(t)$, $j = 1, \cdots, \nu$ *with the following properties*:

1. *For* $a \leq t_0 \leq b$, $\lambda_j(t)$ *can be represented by a series of powers of* $(t - t_0)$ *which converges in a suitable neighborhood of* t_0.
2. λ_0 *is an eigenvalue of* $A(t_0)$ *if, and only if,* $\lambda_0 = \lambda_j(t_0)$ *for some* j.

Once the existence of the analytic eigenvalues has been established, we shall show that it is possible to define associated analytic eigenvectors. In the final

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