ASYMPTOTIC FORMULAE FOR POINT LATTICES OF BOUNDED DETERMINANT AND SUBSPACES OF BOUNDED HEIGHT

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1. Introduction. We are going to study point lattices Λ^k in a Euclidean space E^n. By definition, a k-dimensional point lattice Λ^k $(0 \leq k \leq n)$ is a discrete subgroup of the additive group of E^n, which spans a k-dimensional subspace E^k of E^n. Usually only lattices Λ^n of E^n are studied, and their determinant $d(\Lambda^n)$ is well defined. In the general case, if $k \geq 1$, $d(\Lambda^k)$ is simply defined as the determinant of Λ^k considered as a lattice in its space E^k. Set $d(\Lambda^0) = 1$.

Now assume a Cartesian coordinate system in E^n to be given. Let Γ_0^n be the fundamental lattice of E^n, i.e. the lattice of vectors with integral coordinates. Lattices contained in Γ_0^n will be called *integral lattices* and denoted by Γ, Γ^k, Δ, $\cdots$; general lattices will be denoted by Λ, Λ^k, $\cdots$. A lattice Γ^k is called *primitive* if there is no lattice Γ^{*k} of dimension k properly containing Γ^k. This generalizes the notion of a primitive lattice point.

There is a 1-1 correspondence between k-dimensional subspaces $S^k \subset E^n$ defined over the rationals (defined by linear equations with rational coefficients) and primitive lattices Γ^k. Namely, S^k contains a unique primitive lattice Γ^k, and Γ^k spans S^k. The height of S^k is defined by $H(S^k) = d(\Gamma^k)$. This is a special case of the definition given in [6] for the height of subspaces defined over an algebraic number field.

Write $L(n, k, H)$ for the number of lattices Γ^k of E^n of determinant $\leq H$, $P(n, k, H)$ for the number of primitive lattices Γ^k of E^n of determinant $\leq H$. $P(n, k, H)$ may also be interpreted as the number of rational subspaces S^k of E^n whose height does not exceed H. $V(l)$ where $l = 1, 2, \cdots$ will mean the volume of the unit-ball in E^l, ζ the Riemann zeta-function.

Theorem 1. *For fixed* k, n, $1 \leq k \leq n - 1$,

$$P(n, k, H) = a(n, k)H^n + O(H^{n-b(n,k)}) \tag{1}$$

where

$$a(n, k) = \frac{1}{n}\binom{n}{k}\frac{V(n)V(n-1)\cdots V(n-k+1)}{V(1)V(2)\cdots V(k)} \cdot \frac{\zeta(2)\zeta(3)\cdots\zeta(k)}{\zeta(n)\zeta(n-1)\cdots\zeta(n-k+1)} \tag{2}$$

(*the empty product* $\zeta(2)\cdots\zeta(k)$ *for* $k = 1$ *should mean* 1), *and where*

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