EULERIAN NUMBERS OF HIGHER ORDER

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1. Introduction. Put [2]

$$\left(\frac{1-\lambda}{e^x-\lambda}\right)^s = \sum_{n=0}^{\infty} H_n^{(s)}(\lambda)\frac{x^n}{n!} \qquad (s \geq 1, \quad \text{integer}) \tag{1.1}$$

so that [2; 422]

$$H_n^{(s)}(\lambda) = \sum_{r=1}^{n} (\lambda - 1)^{-r} s(s+1) \cdots (s+r-1) S_2(n, r), \tag{1.2}$$

where $S_2(n, r)$ is the Stirling number of the second kind.

If we put

$$A_n^{(s)}(\lambda) = (\lambda - 1)^n H_n^{(s)}(\lambda),$$

it is immediate from (1.2) that

$$A_n^{(s)}(\lambda) = \sum_{r=1}^{n} (\lambda - 1)^{n-r} s(s+1) \cdots (s+r-1) S_2(n, r) \tag{1.3}$$

is a polynomial of degree $n - 1$ in λ with integral coefficients, say

$$A_n^{(s)}(\lambda) = \sum_{k=1}^{n} A_s(n, k)\lambda^{n-k}. \tag{1.4}$$

For $s = 1$, the numbers $A_1(n, k) \equiv A(n, k)$ are the well-known Eulerian numbers [1, 3, 4, 5] defined by any of

$$A(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k - j)^n, \tag{1.5}$$

$$A(n, k) = kA(n - 1, k) + (n + 1 - k)A(n - 1, k - 1) \tag{1.6}$$

$$\text{with} \quad A(1, k) = \delta_{1,k} \qquad \text{(Kronecker delta)},$$

$$x^n = \sum_{k=1}^{n} \binom{x + k - 1}{n} A(n, k). \tag{1.7}$$

Also we mention the familiar symmetry property

$$A(n, k) = A(n, n + 1 - k). \tag{1.8}$$

The Eulerian numbers have a rather simple combinatorial interpretation [3], [4]. Let $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(n))$ be a permutation on $Z_n = \{1, 2, \cdots, n\}$.

Received February 10, 1967. Supported in part by National Science Foundation grant GP-6382.