A GENERALIZED HILBERT SPACE

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1. Purpose of this article is to generalize the theory of Goldstine and Horwitz [2] to the case when the algebra $\mathfrak{A}$ of scalars is infinite-dimensional. These authors generalized the concept of a Hilbert space by considering a right module H, over a real finite dimensional algebra $\mathfrak{A}$ with an involution, possessing a generalized inner product (,) on H with values in $\mathfrak{A}$. The present author realized that the algebra $\mathfrak{A}$ is in fact a finite-dimensional (real) H^*-algebra and that this theory can be generalized to the case when $\mathfrak{A}$ is an arbitrary proper H^*-algebra (there is no restriction whatsoever on the dimensionality of $\mathfrak{A}$). In the sequel we study a right module H over an arbitrary proper H^*-algebra.

If the algebra of scalars $\mathfrak{A}$ is finite-dimensional, then we have certain simplifications in the theory since in this case $\mathfrak{A}$ possesses an identity, from which it follows immediately that H is also an ordinary Hilbert space. In the general case we have to postulate Schwarz inequality for the derived scalar product (Property 5 in Definition 1 below). It turns out that (in the presence of other axioms) Property 5 in the definition of the Hilbert module H is equivalent to the fact that H is a (right) module over the algebra $\mathfrak{A}_e$ which is obtained from $\mathfrak{A}$ through adjoining an identity to it.

The author succeeded in proving generalizations of many classical theroems in the theory of Hilbert spaces. He found also the most general example of the Hilbert module (Example 3 and Theorem 6 below). In the next section we outline the general theory of H^*-algebras and corresponding trace algebras by stating those facts which will be needed in the other sections.

2. A proper H^*-algebra is a Banach algebra $\mathfrak{A}$ whose underlying Banach space is a Hilbert space and which has an involution $x \rightarrow x^*$ such that $(xy, z) = (y, x^*z) = (x, zy^*)$ for all $x, y, z \in \mathfrak{A}$. A projection is a self-adjoint idempotent member of $\mathfrak{A}$. It was shown in [4, Theorem 4] that for each $a \in \mathfrak{A}$ there exists a sequence $\{e_n\}$ of mutually orthogonal projections in $\mathfrak{A}$ and a sequence $\{\mu_n\}$ of non-negative real numbers such that $a^*a = \sum_{n=1}^{\infty} \mu_n^2 e_n$ (and $a^*ae_n = e_na^*a = \mu_n^2 e_n$ for each n). We define $[a] = \sum_{n=1}^{\infty} \mu_n e_n$. We shall refer to representations $a^*a = \sum_{n=1}^{\infty} \mu_n^2 e_n$, $[a] = \sum_{n=1}^{\infty} \mu_n e_n$ as spectral representations. The element $[a]$ is a unique self-adjoint positive member of $\mathfrak{A}$ such that $[a]^2 = a^*a$ ("$[a]$ is positive" means that $([a]x, x) \geq 0$ for all $x \in \mathfrak{A}$)].

The trace class associated with $\mathfrak{A}$ is the set $\tau(\mathfrak{A}) = \mathfrak{A}^2 = \{xy \mid x, y \in \mathfrak{A}\}$. It has a trace tr( ) defined on it such that $\mathrm{tr}(ab) = \mathrm{tr}(ba) = (a, b^*)$ for all

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