SOME MEASURE-THEORETIC ASPECTS OF RANGE-EQUIVALENCE

By Allen Reiter

1. Introduction. Let X and Y each denote the closed unit interval, and let $f(X) = Y$ be a continuous function. A mapping g is said to be range-equivalent to f (R-equivalent, or simply $g \sim f(R)$,) if there exists an order-preserving homeomorphism ϕ of Y onto itself such that $g = \phi \circ f$. In [2], the author has characterized the R-equivalence classes of C^k functions. Here we consider various other R-equivalence classes, leading to a characterization of the R-equivalence class of absolutely continuous functions.

2. Preliminaries.

2.1. We shall denote by m the Lebesgue measure on the real line.

2.2. If f is a function from X into Y, and $y \in Y$, then $f^{-1}(y)$ will denote the set of all $x \in X$ such that $f(x) = y$.

2.3. Let $f(X) = Y$ be a mapping. Denote by s_f the "counting" function on Y; i.e.

(2.3-1) $s_f(y) =$ "number of points (finite or infinite) in $f^{-1}(y)$". By [3; IX, Theorem 6.4], s_f is a measurable function. We shall also define a counting function S_f on the domain X; i.e., for all $x \in X$, we define

$$S_f(x) = s_f(f(x)). \tag{2.3-2}$$

It is not hard to verify that S_f is also a measurable function.

2.4. The following theorem is due to Nina Bary [1; 635, Theorem III].

THEOREM 1. *Let $f(X) = Y$ be a continuous function. In order for f to be R-equivalent to a function of bounded variation it is necessary and sufficient that every open interval $J \subset Y$ intersect the set of points on which s_f is finite in an uncountable set.*

2.5. A function f is said to satisfy Lusin's condition (N) if f takes sets of measure zero into sets of measure zero. This condition plays a fundamental role in the theory of absolutely continuous functions. Every AC function satisfies condition N; the converse is generally false. If however f is a continuous function of bounded variation defined on a compact set, then the fact that f

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