THE ELEMENTARY THEORY OF RECURSIVELY ENUMERABLE SETS

By A. H. Lachlan

This paper continues the author's work [3] on the lattice of r.e. sets. Let R denote the lattice of r.e. sets, and let $A(R)$ denote the Boolean algebra generated by R whose elements are finite unions of differences of r.e. sets. Denote the quotients of R, $A(R)$ by the ideal of finite sets by R^* (called the lattice of r.e. sets modulo finite sets), $A(R^*)$ respectively. We consider the first order language which has function symbols $\cup$, $\cap$, $'$ to be interpreted as union, intersection, complement respectively and which has just two unary predicate constants E, L. We ask what sentences are true when the quantifiers range over R^* and when $E(x)$, $L(x)$ are interpreted as $x = \emptyset$ the class of finite sets, $x \in R^*$ respectively.

The paper is devoted to giving a decision procedure for $\forall\exists$-sentences. The method is briefly as follows. We consider finite sublattices C of R which are closed in the sense that any r.e. set which can be generated from the members of C by Boolean operations is in C, and such that the subalgebra of $A(R)$ generated by C has no finite atoms. We define a notion of *characteristic* for these sublattices which well-orders them. Next we use the well known result: if α, β are r.e. sets, there exist r.e. subsets α_1, β_1 of α, β respectively such that $\alpha_1 \cup \beta_1 = \alpha \cup \beta$ and $\alpha_1 \cap \beta_1 = \emptyset$. Call a sublattice C of R *separated* if this theorem is satisfied within C. We show that every sublattice C can be imbedded in a separated sublattice C_1 of R such that the characteristic of C_1 is less than or equal the characteristic of C. It follows that we need only consider sentences of the form $(\forall \mathbf{x})(\exists \mathbf{y})[D(\mathbf{x}) \supset P(\mathbf{x}, \mathbf{y})]$, where $D(\mathbf{x})$, $P(\mathbf{x}, \mathbf{y})$ are quantifierless formulas containing just the variables displayed, and where $D(\boldsymbol{\alpha})$ says essentially that the sublattice of R generated by the r.e. sets $\boldsymbol{\alpha}$ is of a particular separated isomorphism type. Such a sentence is called *separated* and its characteristic is defined to be the characteristic of the corresponding isomorphism type. Next by using the existence of a maximal set, a strengthening of Friedberg's Splitting Theorem for a non-recursive r.e. set, and a strengthening of the major subset construction, we construct counterexamples which imply the falsity of a certain recursive class of separated sentences. Finally, in Theorem 4 we give a method whereby the decision problem for any separated $\forall\exists$-sentence not ruled out by one of the counterexamples can be reduced to the decision problem for $\forall\exists$-sentences of lesser characteristic. The proof of Theorem 4 extends the method by which we proved [3, Theorem 2] that for any pair of r.e. sets α, β with $\alpha \subseteq \beta$ and α *hh*-simple in β, $\alpha \cup \beta'$ is r.e. The proof in [3] used index

Received August 8, 1966.