EXTENSIONS OF POSITIVE WEAK*-CONTINUOUS FUNCTIONALS

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1. Introduction. Suppose that A is a subspace of $C(X)$, the space of all continuous complex-valued functions on a compact Hausdorff space, and suppose that on A we are given a positive linear functional ϕ. Let us assume that A contains the constant functions, so that the positivity of ϕ can be formulated: $||\phi|| = \phi(1) = 1$. It is a famous theorem that ϕ can be extended to a positive linear functional on $C(X)$.

We are concerned with the circumstances under which ϕ has a positive extension such that the corresponding measure is absolutely continuous with respect to some given measure m on the space X. If we are to find such an extension, then, one way or another, we must assume that ϕ can be represented as integration against a function in $L^1(m)$. For example, we might disguise the representability slightly, by assuming that ϕ is continuous with respect to the weak topology induced on A by the space $L^1(m)$. But, even if we superimpose on positivity the hypothesis that ϕ is representable by an $L^1(m)$ function, it does not often happen that we can choose that function to be positive.

In this paper, we shall consider several related cases where this extension problem has arisen before, and we shall solve the problem in those cases. Most interesting we feel is the following: Suppose that A is a subalgebra of $C(X)$ and that the functional ϕ is multiplicative on A:

$$\phi(fg) = \phi(f)\phi(g), \qquad f, g \in A.$$

If ϕ can be represented by a complex measure absolutely continuous with respect to m, then ϕ can be represented by a positive measure absolutely continuous with respect to m. In the proof, a certain convex cone plays an essential role. The cone seems to be inherently interesting in the study of various function algebra questions, essentially those which are concerned with generalized Hardy spaces. In particular, the main result of the author's recent paper [5] (proved independently by König [6]) becomes a trivial observation about that cone, once one has shown that the cone is weak* closed in $L^\infty(m)$.

We also consider the case in which A is the space of harmonic functions with continuous boundary values on a domain in R^n. If m is a measure on the interior of the domain such that its support set is "dominating" (see (2.2)), then ϕ is representable by a positive $L^1(m)$ function if and only if ϕ is representable by integration against a (strictly) positive lower semi-continuous function on the boundary of the domain.

We shall be preoccupied with the space $L^\infty(m)$. Evidently, nothing would be lost if we assumed from the outset that A was a subspace of $L^\infty(m)$. We

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