# CHARACTERIZATIONS OF SOME FUNCTION LATTICES

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**Introduction.** The set $C(X, R)$ of all real-valued continuous functions on a topological space has been characterized from a variety of points of view. For $X$ compact, Heider [4] and Anderson and Blair [1], [2] have characterized $C(X, R)$ as a lattice. The primary purpose of this paper is to give a lattice characterization of $C(X, R)$ for $X$ an arbitrary completely regular space. The set $C(X, \bar{R})$ of all extended real-valued continuous functions on a completely regular space is also characterized as a lattice. The results are similar to that of Heider [4].

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**1. Preliminary remarks.** Definitions, notational conventions, and facts essential to the rest of the paper are listed in this section.

All topological spaces are assumed to be completely regular and $X$ will always denote such a space. The Stone–Čech compactification of $X$ is denoted by $\beta(X)$, and the Hewitt realcompactification of $X$ is denoted by $\nu(X)$ [3]. If $K$ is a chain and if $L_1$ and $L_2$ are lattices that contain (a copy of) $K$ as a sublattice, then a *$K$-homomorphism* $\varphi$ of $L_1$ into $L_2$ is a lattice homomorphism such that $\varphi(\alpha) = \alpha$ for all $\alpha \in K$. A *$K$-isomorphism* of $L_1$ into $L_2$ is a bijective $K$-homomorphism. The chain of all constant $K$-valued functions of the space $X$ into $K$ will be identified with the chain $K$, and so $K$ may be considered as a sublattice of $C(X, K)$; in particular, if $\alpha \in K$, then $\alpha$ will denote the constant function on $X$ with value $\alpha$.

If $K$ is a chain and if $\bar{K}$ is the completion of $K$ by cuts, then $\bar{K}$ is compact. Thus, $f \in C(X, K)$ implies that $f$ has a unique continuous extension $f^\beta$ from $\beta(X)$ into $\bar{K}$ [3, 6.5]. It is not hard to see that $f \to f^\beta$ is a $K$-isomorphism of $C(X, K)$ onto a sublattice of $C(\beta(X), \bar{K})$. The image of $C(X, K)$ under this isomorphism will be denoted by $C^\beta(X, K)$. Note that if $f \in C(\beta(X), \bar{K})$, then $f \in C^\beta(X, K)$ if and only if $f(x) \in K$ for all $x \in X$. If $K$ is the chain $R$, each $f \in C(X, R)$ has a unique continuous extension $f^\nu$ from $\nu(X)$ into $R$ [3, 8.6]. It is clear that $f^\nu = f^\beta \mid \nu(X)$ and that $f \to f^\nu$ is an $R$-isomorphism of $C(X, R)$ onto $C(\nu(X), R)$.

If $K$ is a chain, if $L$ is a sublattice of $C(X, K)$ that contains $K$ as a sublattice, and if $p \in \beta(X)$, then the map $\varphi$ defined by

$$\varphi(f) = f^\beta(p) \qquad (f \in L)$$

is a $K$-homomorphism of $L$ into $\bar{K}$; we call $\varphi$ the *$K$-homomorphism of $L$ into*

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