INTRINSIC CHARACTERIZATION OF REGIONS BOUNDED BY CLOSED CURVES

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Regions bounded by closed curves are of fundamental importance in classical analysis, especially complex function theory, and properties concerning the behavior of these regions at boundary points are needed in a variety of contexts. This subject has naturally aroused considerable interest and has been studied in depth for over half a century. Nevertheless, many of the results remain scattered about in the literature, with proofs drawing on rather specialized topological techniques.

We propose here to approach certain questions in this area from a quite different direction, through the use of boundary properties of the Riemann mapping function. Given the Riemann mapping theorem (which, after all, is part of the stock in trade of the complex analyst and a standard topic in courses on complex function theory), the derivations which we present are essentially self-contained. Aside from the Jordan separation theorem for polygonal arcs, only a rudimentary knowledge of topology is presumed. Moreover, the arguments are thoroughly elementary, a fact which must be regarded as yet another tribute to the power of the Riemann mapping theorem.

It should be emphasized that, although our methods of proof are new, the topological results themselves are well known. A brief historical commentary, with references to original papers, is appended.

The problem on which we fix our attention is to characterize those bounded simply connected plane regions Ω for which $\partial\Omega$ is parametrizable as a closed curve, and to do this in terms of the behavior of the region at each of its boundary points. Here a *region* is any nonempty connected open set, and the statement that $\partial\Omega$ *is parametrizable as a closed curve* means that there exists a continuous function Z mapping the real interval $[0, 1]$ onto $\partial\Omega$ with $Z(0) = Z(1)$. We shall denote the neighborhood of radius r about z by $N_r(z)$ and its circumference by $C_r(z)$.

One boundary property that can be used is local accessibility: a point ζ of $\partial\Omega$ is said to be *locally accessible* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that any point of $\Omega \cap N_\delta(\zeta)$ can be joined to ζ by an arc lying (except for its terminal point ζ) in $\Omega \cap N_\epsilon(\zeta)$. A second boundary property of interest is local sequential accessibility: a point ζ of $\partial\Omega$ will be called *locally sequentially accessible* if, for each sequence $\{\zeta_n\}$ of points of Ω converging to ζ and each $\rho > 0$, the open set $\Omega \cap N_\rho(\zeta)$ has a component containing infinitely many ζ_n. Under our hypothesis that Ω is bounded and simply connected, the Riemann mapping theorem asserts the existence of a function χ mapping the unit disc ω conformally onto Ω. We shall refer to χ as a *Riemann mapping function* for Ω.

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