CLOSED SETS OF ALGEBRAIC NUMBERS

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0. Introduction. In the classical literature a PV-number is defined as a real algebraic integer having absolute value greater than 1, and whose remaining conjugates have absolute value less than 1. Salem was the first to prove that the PV-numbers form a closed set.

In this paper we describe a class of adeles which depend on a fixed algebraic number field K, a finite set of valuations V on K containing all Archimedean valuations, and on an element $r \in K$. The adeles are defined to have algebraic (over K) v-components for all $v \in V$, with v-valuation greater than $|r|_v$, and arbitrary v-components for all $v \notin V$, but with v-valuation less than or equal to 1. For each $v \in V$ the remaining conjugates have v-valuation less than $|r|_v$ if v is an Archimedean valuation (less than or equal $|r|_v$ if v is a non-Archimedean valuation). By imposing additional conditions one obtains various closed subsets of this set of adeles (Theorem 3.1). A new version of a theorem due to Kelly appears in Theorem 3.2. Finally a sufficient condition is given for an adele to belong to the derived set of one of the closed subsets.

1. Preliminaries. Let K be a field with a discrete non-Archimedean valuation v, and let K_v denote the completion of K relative to v, Ω_v the algebraic closure of K_v, and $\mathbf{\Omega}_v$ the completion of Ω_v. Let $s > 1$ be chosen so that $\log_s |a|_v$ is an integer for all $a \in K$, and that $\log_s |a|_v = 1$ for some $a \in K$.

Let $A(x) = a_0 + a_1x + \cdots + a_mx^m \in K_v[x]$ with $a_0 \cdot a_m \neq 0$. If we set $y = a_\mu x^\mu$, $a_\mu \neq 0$, then

$$\log_s |y|_v = \mu \log_s |x|_v + \log_s |a_\mu|_v , \qquad 0 \leq \mu \leq m.$$

The points $(\log_s |x|_v , \log_s |y|_v)$ then lie on the straight line

$$\log_s Y = \mu \log_s X + \log_s |a_\mu|_v$$

in the plane with $\log_s X$ and $\log_s Y$ as co-ordinate axes. In fact these points form a discrete subset of the line because the valuation group of K_v is discrete. With each non-zero coefficient of $A(x)$, one can thus associate a line in the $\log_s X - \log_s Y$ plane. The set of points lying above all of these lines is a convex set whose boundary is a polygonal line. If the residue class field of K is sufficiently large, then this line is the graph of $\log_s \max_{|x|_v=r} |A(x)|_v$, as a function of $\log r$, for those r which lie in the valuation group. We shall call this polygonal line the Newton diagram of $A(x)$, though this name is usually used for the dual polygonal line.

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