THE EXISTENCE OF IDEMPOTENTS IN CERTAIN OPERATOR ALGEBRAS

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1. Introduction. In statistical problems of constructing tests invariant under the action of a group via the Hunt–Stein theory [2] one considers the group as a group of linear transformations on an L_∞ space. The usual treatment of the problem constructs an invariant function contained in a certain convex hull, convexity being required in order that certain linear inequalities will be satisfied. The duality of L_∞ to L_1 is used to obtain compactness, hence existence.

Since the statistician is interested only in the existence of an invariant function, consideration has not been given in the literature to the question whether there exist projection operators which are related to the invariant functions obtained. In the sequel we state and prove four results about this question.

We formulate the problem as follows. Suppose E is a B-space with dual E^*. Let $B(E^*, E^*)$ be the B-space of bounded linear transformations of E^* to E^*. Let $C' \subset B(E^*, E^*)$. In the situations considered here, C' will be a convex set which is also a semigroup under the multiplication of operators. In the weak $*$-operator topology, defined below, the closure C of C' is a compact convex semigroup under the operator multiplication. We consider various assumptions which imply the existence of $S_0 \in C$ satisfying, if $T \in C$ then $TS_0 = S_0$.

We will use three topologies on $B(E^*, E^*)$. The *weak $*$-operator topology* has as a base of neighborhoods of zero all sets of the form

$$\{T \mid T \in B(E^*, E^*) \text{ and } |(Ty^*)(x)| < \epsilon, x \in A, y^* \in A^*\},$$

taken over all choices of $\epsilon > 0$, $A \subset E$ and $A^* \subset E^*$, A and A^* finite sets. It is easily verified by use of the Tychonoff product theorem that the unit ball of $B(E^*, E^*)$ is compact in the weak $*$-operator topology.

In addition we will use the *weak operator topology* and the *strong operator topology* on $B(E^*, E^*)$. See [1; pages 475-476]. A base of neighborhoods of zero in the weak operator topology consists of all sets

$$\{T \mid T \in B(E^*, E^*) \quad \text{and} \quad |x^{**}(Ty^*)| < \epsilon, x^{**} \in A^{**}, y^* \in A^*\},$$

taken over all choices of $\epsilon > 0$, $A^{**} \subset E^{**}$, $A \subset E^*$, where A^{**} and A^* are finite sets. In the case that E is a reflexive B-space, then it follows at once that since E and E^{**} act on E^* isomorphically, the base of neighborhoods of zero in the weak $*$-operator topology and the weak operator topology are the same. In the sequel we use the compactness of the unit ball of $B(E^*, E^*)$ in the weak operator topology when E is a reflexive B-space.

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