NOTE ON A PAPER BY M. F. TINSLEY

By Alan Zame

In a recent paper [1], M. F. Tinsley proves the following result: Suppose $d_1, \cdots, d_r$ are elements of an (additive) abelian group G of order n, the d_i being distinct and $r \geq 2$. Suppose that the non-negative integers $x_1, \cdots, x_r$ are a solution of the equation

$$(*) \qquad x_1 d_1 + \cdots + x_r d_r = \theta$$

(where θ is the identity of G), at least two of the x_i are positive and that $x_1 + \cdots + x_r \geq n$. Then there exist non-negative integers $y_1, \cdots, y_r$, with each $y_i \leq x_i$ and at least one $0 < y_i < x_i$ such that

$$y_1 d_1 + \cdots + y_r d_r = \theta.$$

In other words, "primitive" solutions of (*) must have $x_1 + \cdots + x_r < n$. Tinsley's proof of this interesting result is rather complicated, so we offer the following simple proof of a slightly more general result.

Let G be any group of order $n > 2$ (written multiplicatively) and let $g_1, \cdots, g_{n-1}$ be any (not necessarily distinct) elements of G with, say, $g_1 \neq g_2$. Then some product $g_{\alpha_1} \cdots g_{\alpha_r}$ of these g_i, where the α_i are distinct, is the identity e of G.

Proof. If either g_1 or g_2 is e, the result is trivial; otherwise, the elements g_1, g_2 and $g_1 g_2$ are distinct. Consider the n terms

$$g_3, g_3 g_4, \cdots, g_3 g_4 \cdots g_{n-1} = g, g g_1, g g_2, g g_1 g_2.$$

Since G is of order n, either one of these terms is e, in which case the result follows, or two of these terms are equal. If $g_3 \cdots g_k = g_3 \cdots g_r$ $(r > k)$ then $g_{k+1} \cdots g_r = e$. If $g_3 \cdots g_k = g\gamma$ where γ is either g_1 or g_2 or $g_1 g_2$, then $g_{k+1} \cdots g_{n-1}\gamma = e$. But these are the only equalities we can have, so the result follows.

References

1. M. F. Tinsley, *A Combinatorial Theorem in Number Theory*, this Journal, vol. 33(1966), pp. 75–79.

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