CONGRUENCES ON EXTENSIONS OF SEMIGROUPS

By Mario Petrich

If V is a semigroup, S an ideal of V, T a semigroup with zero isomorphic to V/S (the Rees factor semigroup of V modulo S), then V is said to be an ideal extension of S by T (henceforth called an *extension*). These extensions have been studied by Clifford [1] (see also [2, 4.4]), Yoshida [4], Grillet and Petrich [3] when S and T are arbitrary or satisfy certain conditions (the most notable one being that S be weakly reductive). In all the papers mentioned, multiplication in V is expressed in terms of multiplications in S and T and the translational hull of S.

We consider the problem of describing congruences on V in terms of congruences on S and T. To do this for an arbitrary congruence ν on V, a great number of conditions on a pair σ, τ of congruences on S and T, respectively, is required, and they are too strongly reminiscent of the requirement that ν be compatible with the multiplication in V. The general case is thus abandoned in favor of the case when $\sigma(= \nu|_S)$ is weakly reductive (that is, when S/σ is a weakly reductive semigroup). In such a case, it is sufficient for a pair σ, τ of congruences on S and T, respectively, to satisfy certain simple conditions to induce a congruence on V.

In §1 we describe a construction which yields all congruences ν on V such that $\nu|_S$ is weakly reductive. For some classes of semigroups S, this construction yields all congruences on V. Conditions on congruences σ, τ simplify if the extension is given by a partial homomorphism. §2 is devoted to a description of the homomorphic image induced by a congruence ν on V in terms of homomorphic images of S and T induced by σ and τ, respectively, where σ and τ determine ν, and σ is weakly reductive. Here we make use of the construction of extensions due to Clifford [1] and Yoshida [4]. A special case of interest is when V is a strict extension [3] of S. Finally, in §3 we consider an extension V of S by T determined by a partial homomorphism and give a construction of congruences on such V. Here we are able to obtain somewhat larger class of congruences ν than in §1. We describe the homomorphic image induced by ν if $\nu|_S$ is weakly reductive. Some special cases are also of interest here.

Throughout, V denotes an extension of a semigroup S by a semigroup T with zero 0. We consider V as the union $T^* \cup S$, where $T^* = T\backslash 0$. Elements of T^* are denoted by capital letters, while those of S by lower case letters; we consider them simultaneously as elements of either T or S, respectively, or V. In §1 we denote multiplication in all three semigroups by juxtaposition, while in the remaining two sections we reserve a special symbol for multiplication in V.

For any semigroup D, $\mathfrak{C}(D)$ is the set of congruences on D; for $\delta \in \mathfrak{C}(D)$,

Received April 22, 1966.