A CONVERGENCE CRITERION WITH APPLICATIONS TO SERIES IN LOCALLY CONVEX SPACES

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1. Introduction. A convergence criterion for sequences in locally convex Hausdorff spaces, Theorem 2.1, is the main theorem and serves as the main lemma for the remainder of the paper. It is applied to extend a characterization of "weak unconditional convergence" known for Banach spaces [1, Lemma 2] to locally convex Hausdorff spaces. It is also used to establish for locally convex Hausdorff spaces a necessary and sufficient condition for subseries cauchy convergence which has as one of its corollaries a result for Banach spaces proved by Dunford [4, Theorem 75], Pettis [7; 281–282], and Gelfand [5; 244]. The results in this paper on linear transformations determined by series are presented, not merely to extend certain theorems on series of Gelfand [5, Part II, §§4 and 5] from Banach spaces to more general spaces but to show some of the implications for a general theory of series in linear topological spaces resulting from the fact that equicontinuous and strongly bounded sets in the adjoint space may not coincide.

2. Convergence criterion.

Theorem 2.1. *Let $(E, \mathfrak{F})$ be a locally convex Hausdorff space and E^* the space of $\mathfrak{F}$-continuous linear functionals on E. A $w(E, E^*)$-Cauchy sequence $\{x_p\}$ in E is $\mathfrak{F}$-Cauchy if and only if whenever $\{f_n\}$ is an equicontinuous sequence of elements of E^* such that* $\lim_n f_n(x_p) = 0$ *for each $p \in \omega$, it follows that* $\lim_n f_n(x_p) = 0$ *uniformly with respect to $p \in \omega$.*

Proof. Since a proof of the necessity of the condition is straightforward we present only the proof for its sufficiency. Suppose that $\{x_p\}$ is $w(E, E^*)$-Cauchy but is not $\mathfrak{F}$-Cauchy. Thus there exists a closed convex circled $\mathfrak{F}$-neighborhood V of 0 and an increasing sequence p_n of positive integers such that for each n, $s_n \equiv x_{p_{n+1}} - x_{p_n} \notin V$. Then [6, 14.4; 119], for each n, there exists $f_n \in E^*$ such that $f_n(s_n) = 1$ and $\sup \{|f_n(x)| : x \in V\} < 1$. Thus the sequence $\{f_n\}$ is $\mathfrak{F}$-equicontinuous. Being equicontinuous, it is pointwise bounded. The equicontinuity and pointwise boundedness enable us to select a subsequence $\{f_{n_m}\}$ which has a $w(E^*, E)$ cluster point f_0 with the property that $\lim_m f_{n_m}(x_p) = f_0(x_p)$, $p \in \omega$. We describe this selection process in more detail. Since $\{f_n(x_1)\}$ is a bounded sequence of scalars, it has a convergent subsequence which we denote by $\{f_{1,m}(x_1)\}_{m \in \omega}$ with limit r_1. Now the selected subsequence $\{f_{1,m}\}_{m \in \omega}$ is again equicontinuous so there exists a convergent subsequence of $\{f_{1,m}(x_2)\}$.

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