ON THE EXPANSION PROBLEM FOR NONSELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS OF SECOND ORDER

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The increasing use of nonself-adjoint ordinary differential operators of second order in applied mathematics has resulted in the need to extend the theory of such operators. The theory is presently well developed in only a few special cases ([3], [7], [8], [9], [10], [11]), and only two ([7] and [8]) relax hypotheses sufficiently to give some freedom of maneuver. This paper significantly extends the theory by making only mild assumptions which seem to cover most of the cases previously considered.

Let $q(x) = q_1(x) + iq_2(x)$ be a complex function, summable over every closed subinterval of an arbitrary interval (a, b). Let $\lim_{x\to b} q_2(x) = \gamma$, $\lim_{x\to a} q_2(x) = \delta$, where $-\infty \leq \delta \leq \gamma \leq \infty$. We consider the differential expression $ly = -y'' + q(x)y$ on (a, b).

A nonself-adjoint operator L on $L^2(a, b)$ is defined by letting $Ly = ly$ over a suitable domain. The spectrum of L is found to lie on the horizontal lines $\operatorname{im} \lambda = \gamma$, $\lambda = \operatorname{im} \delta$, with an additional countable number of eigenvalues scattered throughout the λ-plane. Finally, L^* is found, and a spectral resolution for L is derived.

1. Solutions of $(l - \lambda)y = 0$ in $L^2(r, b)$. Let r be an arbitrary point in (a, b). Let $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ be solutions of $(l - \lambda)y = 0$ which satisfy $\theta(r, \lambda) = 1$, $\theta'(r, \lambda) = 0$, $\varphi(r, \lambda) = 0$, $\varphi'(r, \lambda) = -1$. Note that $W[\theta, \varphi] = \theta(x, \lambda)\varphi'(x, \lambda) - \theta'(x, \lambda)\varphi(x, \lambda) \equiv -1$.

Theorem 1.1. *If $\lambda = \mu + i\nu$, $\nu \neq \gamma$, then there exists a solution $\psi(x, \lambda)$ of $(l - \lambda)y = 0$ in $L^2(r, b)$.*

Proof. Let λ be fixed with $\nu \neq \gamma$. If γ is finite, we choose s in $[r, b)$ such that $|q_2(x) - \gamma| < |\nu - \gamma|/2$ for almost all x in (s, b). If γ is infinite, we choose s such that $|q_2(x) - \nu| > 1$ for almost all x in (s, b). Note that s is dependent on ν. Let $y_1(x, \lambda) = \theta'(s, \lambda)\varphi(x, \lambda) - \varphi'(s, \lambda)\theta(x, \lambda)$, and $y_2(x, \lambda) = \theta(s, \lambda)\varphi(x, \lambda) - \varphi(s, \lambda)\theta(x, \lambda)$. Then $y_1(s, \lambda) = 1$, $y_1'(s, \lambda) = 0$, $y_2(s, \lambda) = 0$, $y_2'(s, \lambda) = -1$.

If β is in (s, b), let $\psi(x, m, z) = y_1(x, \lambda) + my_2(x, \lambda)$ be a solution of $(l - \lambda)y = 0$ satisfying an arbitrary boundary condition $z\psi(\beta, m, z) + \psi'(\beta, m, z) = 0$. $(m = -(y_1(\beta, \lambda)z + y_1'(\beta, \lambda))/(y_2(\beta, \lambda)z + y_2'(\beta, \lambda)).)$ As z takes on all real values, m describes a circle $C(\beta)$ with center $\rho(\beta) = -W[y_1(\beta, \lambda), \bar{y}_2(\beta, \lambda)]/W[y_2(\beta, \lambda), \bar{y}_2(\beta, \lambda)]$ and radius $r(\beta) = |2 \int_s^\beta [\nu - q_2(x)]\, |y_2(x, \lambda)|^2\, dx|^{-1}$.

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