MAJORIZATION BY UNIVALENT FUNCTIONS

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Let $f(z)$ and $g(z)$ be functions analytic in a set D in the complex plane. We will say that $f(z)$ is majorized by $g(z)$ in D providing $|f(z)| \leq |g(z)|$ for each z in D.

Assume that $f(z)$ is majorized by $g(z)$ in $|z| < 1$ and $g(0) = 0$. Our first result asserts that $\max_{|z|=r} |f'(z)| \leq \max_{|z|=r} |g'(z)|$ for each number r in the interval $0 \leq r \leq \sqrt{2} - 1$. Since we may choose $g(z) = z$, this generalizes the well-known theorem: if $f(z)$ is analytic for $|z| < 1$, $|f(z)| \leq 1$, and $f(0) = 0$, then $|f'(z)| \leq 1$ for $|z| \leq \sqrt{2} - 1$ [1; 19]. There is no positive constant r, independent of $f(z)$ and $g(z)$, such that $f'(z)$ is majorized by $g'(z)$ in $|z| \leq r$. This can be shown by the example $f(z) = z^3 - 2rz^2$ and $g(z) = z^2 - 2rz$, since $g'(r) = 0$ and $f'(r) \neq 0$ for $r \neq 0$. The situation is different if $g(z)$ is univalent, an assumption which implies that $g'(z) \neq 0$. We show that if $g(z)$ is univalent then $f'(z)$ is majorized by $g'(z)$ in $|z| \leq 2 - \sqrt{3}$. Also, if $g(z)$ maps $|z| < 1$ one-to-one onto a convex domain, then $f'(z)$ is majorized by $g'(z)$ in $|z| \leq \frac{1}{3}$. These results are "best possible".

The second part of this paper concerns the problem of finding upper bounds on $|a_n|$ where $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is majorized by a univalent function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $|z| < 1$. We prove that $|a_n| \leq n$ for $n = 1, 2, 3$ and conjecture that $|a_n| \leq n$ for all n. This conjecture implies the "Bieberbach conjecture" that $|b_n| \leq n$, since $f(z) = g(z)$ is possible under the hypotheses. The proof of Theorem 2 indicates that the inequality $|a_n| \leq n$ is a consequence of the assertion : if the odd analytic function $h(z) = z + \sum_{k=1}^{\infty} c_{2k+1} z^{2k+1}$ is univalent for $|z| < 1$, then

$$(1) \qquad 1 + |c_3|^2 + |c_5|^2 + \cdots + |c_{2n-1}|^2 \leq n.$$

In [6] the conjecture (1) was posed, and it was shown that it implies the Bieberbach conjecture.

We establish the inequality $|a_n| \leq n$ for all n, if $g(z)$ is spiral-like for $|z| < 1$. This assumes the existence of a real number α, $|\alpha| < \pi/2$, such that $\text{Re}\,\{e^{i\alpha} zg'(z)/g(z)\} > 0$ for $|z| < 1$ [8]. The special case $\alpha = 0$ corresponds to univalent functions mapping $|z| < 1$ onto domains starlike with respect to $w = 0$. The inequality $|a_n| \leq n$ is also obtained in the case $g(z)$ is typically real. This assumes $g(z)$ is analytic in $|z| < 1$ and is real if and only if z is real [7]. This class of typically real functions includes the class of functions univalent in $|z| < 1$ with real coefficients. We also show that $|a_n| < en$ where $g(z)$ is an arbitrary univalent function, and for univalent functions $g(z)$ mapping

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