ON THE CESARI-CAVALIERI AREA OF A SURFACE

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1. Introduction. In his book, *Surface Area* (Princeton, 1956) [1, Chapter VI, §20], Lamberto Cesari proved an inequality which is of basic importance in the theory of Fréchet surfaces and of variational problems connected with them. Since the inequality contained the Cavalieri principle for surfaces as a special case, he called it the Cavalieri inequality. Recently various inequalities of this type have been developed. We therefore shall refer to the inequality of Cesari as the Cesari–Cavalieri inequality.

In [1] the inequality was shown for surfaces defined over a planar disk. In [3], [4] the inequality was extended to surfaces defined over compact two dimensional manifolds. In [5] the inequality was extended to include functions which are of a larger class than those in [1, Chapter VI, §20]. Recently R. E. Fullerton [6] has defined an area functional based on the Cesari–Cavalieri inequality and he has shown that this area functional coincides with the Lebesgue area for smooth non-parametric surfaces.

The area functional as defined by Fullerton depends on a generalization of the Cavalieri principle. Thus the area of a surface is expressed essentially in terms of an integral of the length function of a family of curves which cover the surface.

The concept of a contour, the generalized length, and the Cesari–Cavalieri inequality are introduced in the next section. The final section is devoted to proving the coincidence of the Lebesgue area with the Cesari–Cavalieri area, as defined by Fullerton, for smooth parametric surfaces. This is the principal result of the present paper.

The more general question as to whether the area as defined by Fullerton coincides with the Lebesgue area for all continuous parametric surfaces without smoothness conditions is thus far unanswered. The general question cannot be approached by the methods of this paper which make essential use of the smoothness hypothesis.

2. Definitions and preliminary results. Let Q be a two-dimensional compact complex and let $T : Q \to E_N$ be a continuous mapping of Q into N-dimensional Euclidean space ($N \geq 2$). Then (T, Q) define a Fréchet surface S. Let $[S]$ be the set of points in E_N occupied by the surface S. Let $f : [S] \to$ Reals be a

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