NOTE ON A PAPER OF CHEEMA AND GORDON

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In [1] Cheema and Gordon give an ingenious combinatorial proof for the generating function of two-line partitions, and obtain certain congruences modulo 5 and 3 for two- and three-line partitions. They go on to assert that these congruences can be interpreted combinatorially in terms of ranks:

"To do this let τ be any k-line partition and let $\delta(\tau)$ be the greatest part of τ minus the number of parts on the first row of τ. It then turns out that for $n \equiv 3$ or $4 \pmod 5$, the residues (mod 5) of the numbers $\delta(\tau)$, where τ runs through all 2-line partitions of n, are equidistributed among the five residue classes 0, 1, 2, 3, 4. The proof is rather complicated and will not be gone into here."

In fact, a machine-computed table, using Macmahon's generating functions, shows that this is true for $n = 3, 4, 8$, or 9, and for no other values of n less than 100. I have also verified its falsehood independently by hand, using the combinatorial definition, for $n = 13$ and $n = 14$.

Their analogous assertion for $n \equiv 2 \pmod 3$ and 3-line partitions is true for $n = 2, 5, 8, 11, 20, 26$ and no other relevant value of n less than 60. However, although the proof to which they refer must contain some error, the idea of extending the rank definition to the k-line case seems to be significant, since the rank differences in the tables are very much smaller than one might expect.

Writing

$$f(y) = \prod_{\nu=1}^{\infty} (1 - y^{\nu}),$$

$$\sum_{n=0}^{\infty} p_{-2}(n) y^n = 1/f^2(y),$$

Cheema and Gordon state and prove the formula

$$\sum_{n=0}^{\infty} p_{-2}(5n + 3) y^n = 10 f^4(y^5)/f^6(y) + 125 y f^{10}(y^5)/f^{12}(y). \tag{1}$$

It is worth noting that one can obtain formulae of a related kind for the series $\sum_{n=0}^{\infty} p_{-2}(5n + 4) y^n$ and $\sum_{n=0}^{\infty} p_{-2}(5n + 2) y^n$; these are of course more involved than (1) which essentially belongs to the subgroup $\Gamma_0(5)$ of the modular group, whereas the formulae below belong to $\Gamma(5)$. However they can be proved quite easily by the method of division of periods of theta-functions which is used in [1] to obtain (1). We have, for $|y| < 1$,

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