A MINKOWSKIAN TYPE BOUND FOR A CLASS OF RELATIVE QUADRATIC FIELDS

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1. Introduction. Let K_1 denote a totally real field of degree n over the rationals with class number 1. Let $\mu_0 = \tau^2 - 4\epsilon$ be a totally negative integer in K_1 where τ and ϵ (unit) are integers in K_1. Then $K_2 = K_1(\sqrt{\mu_0})$ will be called a fixed point field. This name is used since μ_0 arises from fixed points of the Hilbert modular group for K_1 [1; 197]. By choosing $\tau = 0$ and $\epsilon = 1$, we note that $K_1(\sqrt{-1})$ is always a fixed point field.

In this paper a Minkowskian type bound is obtained for fixed point fields using the following theorem due to H. Cohn [1; 201].

THEOREM 1. *Let τ and ϵ (unit) be integers in K_1 and let $\mu_0 = \tau^2 - 4\epsilon$ be totally negative. Then the solvability of*

$$\xi^2 - \tau\xi + \epsilon \equiv 0 \pmod{\gamma}$$

in integers in K_1 implies the solvability of

$$\xi_1^2 - \tau\xi_1\xi_2 + \epsilon\xi_2^2 = \gamma\gamma_1$$

in integers in K_1 with

$$0 < |N(\gamma_1)| \leq N^{\frac{1}{2}}(\mu)/2^n H_1$$

where $H_1 = \inf \{\operatorname{Im} Z \cdot \operatorname{Im} Z' \cdots \operatorname{Im} Z^{(n-1)}: (Z, Z', \cdots, Z^{(n-1)})$ is a fixed point in the fundamental domain of the Hilbert modular group over $K_1\}$.

The assumption that K_1 has class number 1 is made to ensure $H_1 > 0$ as shown by Maass [2]. It is also known [1; 202] that if K_1 is a quadratic field, then $H_1 > 2/d$, where d is the discriminant of K_1.

By taking $n = 1$, $\tau = 0$ and $\epsilon = 1$, Theorem 1 reduces to the classical result of Fermat which states that the solvability of $x^2 + 1 \equiv 0 \pmod{m}$ implies the solvability of $x^2 + y^2 = m$ in rational integers. (H_1 for Q is $\sqrt{3}/2$ and hence $\gamma_1 = 1$.) This classical result in turn can be used to prove $Q(\sqrt{-1})$ has class number 1. The theorem below, giving the Minkowskian type bound for fixed point fields, also yields as a special case, the result that $Q(\sqrt{-1})$ has class number 1.

2. Minkowskian type bound.

THEOREM 2. *Every ideal class of a fixed point field contains an ideal of norm less than or equal to $N^{1/2}(\mu_0)/2^n H_1$.*

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