THE NUMBER OF SQUAREFREE DIVISORS OF AN INTEGER

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Let $\theta(n)$ denote the number of squarefree divisors of n. Mertens proved in 1874 that

$$(1) \qquad \sum_{n \le x} \theta(n) = \frac{x}{\zeta(2)} \left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + O(x^{\frac{1}{2}} \log x),$$

where $\zeta(s)$ is the Riemann zeta function and γ is Euler's constant. Recently Eckford Cohen [1] gave a new proof of (1). In this note, we improve the error term to $O(x^{\frac{1}{2}})$.

We use the following results.

Lemma 1. *If $\tau(n)$ denotes the number of divisors of n, then*

$$\sum_{n \le x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^c),$$

where $c < \frac{1}{2}$.

Actually it is known that $c < \frac{1}{3}$ (cf. [4]). There is a conjecture that $c = \frac{1}{4} + \epsilon$, for any $\epsilon > 0$.

Lemma 2. *If $\mu(n)$ is the Möbius function, then for arbitrary q,*

$$M(x) = \sum_{n \le x} \mu(n) = O(x \log^{-q} x).$$

For a proof, see [3]. An easy deduction from Lemma 2 (or see [2]) is

Lemma 3. *For arbitrary q, we have*

$$\sum_{n \le x} \mu(n) n^{-2} = 1/\zeta(2) + O(x^{-1} \log^{-q} x).$$

Lemma 4. *For arbitrary q,*

$$\sum_{n \le x} \mu(n) n^{-2} \log n = \zeta'(2)/\zeta^2(2) + O(x^{-1} \log^{-q} x).$$

Proof. From $\sum_{n=1}^{\infty} \mu(n) n^{-s} = 1/\zeta(s)$, we have

$$\sum_{n \le x} \mu(n) n^{-2} \log n = \zeta'(2)/\zeta^2(2) - \sum_{n > x} \mu(n) n^{-2} \log n.$$

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