ENUMERATION OF SYMMETRIC ARRAYS

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1. Introduction. Let $H(n, r)$ denote the number of $n \times n$ arrays $[a_{ij}]$, where the a_{ij} are nonnegative integers that satisfy

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = r. \tag{1.1}$$

Anand, Dumir and Gupta [1] have proved that if $A(n) = H(n, 2)/(n!)^2$, then

$$\sum_{n=0}^{\infty} A(n)x^n = (1 - x)^{-\frac{1}{2}}e^{x/2}. \tag{1.2}$$

They have also proved that

$$H(3, r) = \binom{r+2}{2} + 3\binom{r+3}{4}, \tag{1.3}$$

from which it follows that

$$\sum_{r=0}^{\infty} H(3, r)x^r = \frac{1 + x + x^2}{(1 - x)^5}. \tag{1.4}$$

They conjecture that

$$H(n, r) = \sum_{i=0}^{\binom{n-1}{2}} c_i \binom{r + n + i - 1}{n + 2i - 1},$$

where the c_i depend on n alone.

In the present paper we consider an analogous problem for symmetric arrays. Let $S_n(r)$ denote the number of $n \times n$ arrays a_{ij}, where the a_{ij} are integers such that

$$a_{ij} = a_{ji} \geq 0 \qquad (i, j = 1, 2, \cdots, n) \tag{1.5}$$

and

$$\sum_{i=1}^{n} a_{ij} = r \qquad (j = 1, 2, \cdots, n). \tag{1.6}$$

Clearly

$$S_n(0) = 1 \qquad (n = 1, 2, 3, \cdots). \tag{1.7}$$

We shall show that

$$\sum_{n=0}^{\infty} S_n(1) \frac{x^n}{n!} = \exp\left(x + \tfrac{1}{2}x^2\right) \qquad (S_0(1) = 1). \tag{1.8}$$

Received May 11, 1966. Supported in part by NSF Grant GP-5174.