A COMBINATORIAL DISTRIBUTION PROBLEM

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1. Kenji Mano [1] has investigated the number $H(n, r)$ of ways in which n distinct things, $n \geq 1$, each replicated r times, $r \geq 1$, can be distributed in equal numbers among n persons. He gives an intricate formula for the case $r = 2$. Here, we give

(i) Some inequalities for $H(n, r)$, true for all positive n and r;
(ii) A simple recursion formula for $H(n, 2)$; true for $n \geq 1$;
and (iii) A formula for $H(3, r)$; true for all $r \geq 1$.

We obtain also some congruence properties of the functions involved in (ii).

A plausible formula for $H(n, r)$ is stated.

The case where each person gets distinct objects is also considered for $r = 2$. We denote the number of ways of distribution by $H^*(n, r)$ in this case.

Throughout this paper, all small Roman letters denote integers ≥ 0.

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2. Consider the set M_r of $n \times n$ matrices $[a_{ij}]$, such that

$$\sum_{i=1}^{n} a_{ij} = r = \sum_{j=1}^{n} a_{ij}, \qquad a_{ij} \geq 0. \tag{1}$$

If we take a_{ij} to denote the number of articles of the i-th type which are given to the person j, in any distribution, then it is readily seen that $H(n, r)$ is the number of matrices in M_r.

Since the number of solutions of the equation:

$$\sum_{i=1}^{n} x_i = r, \tag{2}$$

in non-negative integers, is

$$\binom{n + r - 1}{r};$$

the number of matrices in M_r cannot exceed

$$\binom{n + r - 1}{r}^{n-1}, \tag{3}$$

because, once the first $(n - 1)$ rows in a matrix of M_r have been completed, the last row can be filled up in only one way, if at all.

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