NORMAL FAMILIES AND COMPLETELY REGULAR SPACES

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This paper is based on the work of Frink [1]. In that paper, Frink gives an internal description of Tychonoff spaces (completely regular and T_1). These spaces are characterized by being T_1 and possessing a normal base for closed sets. A normal base is a base which is a disjunctive ring of sets, disjoint members of which may be separated by disjoint complements of members. The proof involves imbedding the space in a compactification of Wallman type which is shown to be Hausdorff.

Unfortunately, this proof cannot be extended to cover all completely regular spaces. In the attempt to make this extension, we found a somewhat simpler characterization and a more direct proof. The ring property is not necessary.

As Frink points out, since complete regularity is a hereditary property, it is desirable to have a direct internal proof. We will attempt to do this.

A family $\mathfrak{F}$ of closed sets will be called *normal* if any two disjoint members A and B of $\mathfrak{F}$ are contained in disjoint complements C' and D' of members C and D of $\mathfrak{F}$; that is, $A \subset C'$, $B \subset D'$ and $C' \cap D' = \phi$.

A family $\mathfrak{F}$ of closed sets will be called *separating* if it separates points from closed sets; that is, given any closed set S and any point x not in S, there exist sets A and B in $\mathfrak{F}$ such that $x \in A$, $S \subset B$, and $A \cap B = \phi$.

If $\mathfrak{F}$ is a family of sets in a space E, define $\mathcal{C}\mathfrak{F} = \{\mathcal{C}F : F \in \mathfrak{F}\}$, where $\mathcal{C}F = E - F$. A *scale* in a family $\mathfrak{F}$ of closed sets will be a pair of mappings (f, g) of the dyadic rationals $\mathfrak{D}$ of $(0, 1)$ into $\mathcal{C}\mathfrak{F}$ and $\mathfrak{F}$ respectively such that $\alpha < \beta$ implies $f(\alpha) < g(\alpha) < f(\beta)$.

THEOREM 1. *A topological space is completely regular if and only if it possesses a normal separating family of closed sets.*

Proof. It is shown in Gillman and Jerison [2] that the family of all zero sets of continuous real-valued functions defined on a completely regular space is normal and separating. (T_1 is not necessary in their proof.)

Now suppose E is a topological space which possesses a normal separating family $\mathfrak{F}$ of closed sets. Let F be a closed subset of E and x be a point not contained in F. Since $\mathfrak{F}$ is separating, there exist sets A and B in $\mathfrak{F}$ such that $x \in A$, $F \subset B$ and $A \cap B = \phi$.

Since $\mathfrak{F}$ is normal, there exist disjoint sets C' and D' of $\mathcal{C}\mathfrak{F}$ separating A and B. We then have $x \in A \subset C' \subset D \subset B'$, where $B' = \mathcal{C}B$. Now sets G, H in $\mathfrak{F}$ and G', H' in $\mathcal{C}\mathfrak{F}$ may be found such that $A \subset G' \subset G \subset C' \subset D \subset H' \subset H \subset B'$. Define $f(\frac{1}{2}) = C'$, $g(\frac{1}{2}) = D$, $f(\frac{1}{4}) = G'$, $g(\frac{1}{4}) = G$, $f(\frac{3}{4}) = H'$, and $g(\frac{3}{4}) = H$. By induction, it can be seen that a scale (f, g) can be produced in $\mathfrak{F}$ such that

Received January 24, 1966. Supported by NSF Research Grant GP-2214.