AN EXTREMAL PROBLEM IN QUASICONFORMAL MAPPINGS

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1. Introduction. Professor Ahlfors [3] has given a constructive method for obtaining a sequence of piecewise affine mappings of the plane, which converges uniformly in the plane to a given K-quasiconformal mapping of the finite plane onto itself, provided only that $1 \leq K < \sqrt{3}$. In applying this result toward the extension of a plane K-quasiconformal mapping to a quasiconformal mapping of the half space [2], it is further required that the maximal dilatations of the piecewise affine approximations can be made arbitrarily close to 1, by requiring that K be sufficiently near 1. It thus becomes of some interest to find the maximal dilatations which can occur for the approximating mappings obtained from the given method of construction.

Accordingly, for $1 \leq K < \infty$, let Q_K denote the class of K-quasiconformal mappings of the finite plane onto itself. For $\varphi \in Q_K$, $K < \sqrt{3}$, $j = 0, 1, 2, \cdots$, let $\varphi^{(j)}$ denote the j-th approximating mapping to φ as constructed by the Ahlfors method, and let $K(\varphi^{(j)})$ denote the maximal dilatation of $\varphi^{(j)}$. In this paper, we will review the construction of the mappings $\varphi^{(j)}$, in order to calculate

$$\xi(K) = \sup \{K(\varphi^{(j)}) : \varphi \in Q_K, \quad j = 0, 1, 2, \cdots\}$$

for the special case in which the "pieces" of the approximating mappings are equilateral triangles, and in order to discuss extensions of the method to general quasiconformal mappings.

2. Definitions and preliminary results. We select the following definition [6], from the large number of equivalent definitions of quasiconformality: let φ be an orientation preserving homeomorphism of the finite plane onto itself, and set

$$H(z) = \limsup_{r \to 0} \frac{\max_\theta |\varphi(z + re^{i\theta}) - \varphi(z)|}{\min_\theta |\varphi(z + re^{i\theta}) - \varphi(z)|}.$$

Then we say that φ is quasiconformal if $H(z)$ is bounded. If this is the case, the maximal dilatation $K(\varphi)$ is defined by ess sup $H(z)$. φ is said to be $K - qc$, or to be in the class Q_K, if $K(\varphi) \leq K$, where $1 \leq K$.

The affine mapping, $\varphi(z) = pz + q\bar{z}$, is quasiconformal if $|p| > |q|$, and in this case,

$$K(\varphi) = \frac{|p| + |q|}{|p| - |q|}. \tag{2.1}$$

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