SOME CONGRUENCES INVOLVING BINOMIAL COEFFICIENTS

By L. Carlitz

Let K be a field of characteristic p, let

$$A(z) = \sum_{j=0}^{e} A_j z^j \qquad (A_j \in K)$$

be a polynomial over K of degree e and put

$$(1 + z)^b A(z) = E(z) = \sum_{j=0}^{\infty} E_j z^j,$$

where b is a nonnegative integer and $E_j = 0$ for $j > b + e$. Also let n be any positive integer. S. Abhyankar [1], [2] proved the following result.

Theorem A. *If either*

(i) $$e < p^{n-1} \quad \textit{and} \quad E_1 = E_2 = \cdots = E_{p^{n-1}} = 0$$

or

(ii) $$E_1 = E_2 = \cdots = E_{p^{n-1}+e} = 0,$$

then $b + e \equiv 0 \pmod{p^n}$.

Drazin [3] has proved the following more general result; square brackets as usual denote the greatest integer function.

Theorem B. *If* $E_1 = E_2 = \cdots = E_d = 0$, *where* $d = p^{n-1}(1 + [e/p^{n-1}])$, *then* $b + e \equiv 0 \pmod{p^n}$.

Indeed Drazin proves the following property of binomial coefficients which is easily seen to imply Theorem B.

Theorem C. *If*

$$\binom{-b}{e} \not\equiv 0 \pmod{p}$$

and

$$\binom{-b}{j} \equiv 0 \pmod{p} \qquad (e + 1 \leq j \leq p^{n-1}(1 + [e/p^{n-1}]),$$

then $b + e \equiv 0 \pmod{p^n}$.

It may be of interest to point out that Theorem C can be proved very rapidly as follows. Since it is no more difficult, we shall prove the following slightly more general result.

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