THE INVARIANCE OF DOMAIN UNDER ACYCLIC MAPPINGS

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1. Introduction. The classical theorem of Brouwer concerning the invariance of domain states that if f is a continuous injective mapping of an n-manifold X into an n-manifold Y, then fX is open in Y. The injectivity condition can be somewhat weakened. For instance, it follows rather easily from the general results of Wilder [7] that the same conclusion holds if we replace injectivity by the condition that all point-inverses be compact and homologically trivial. Moreover, X and Y may be allowed to be generalized manifolds. The aim of this note is to give a direct proof for this result. A particular case was proved in an earlier paper [5] of the author. There we made use of the topological index of a mapping. This concept cannot be used if the manifolds in question are non-orientable. Therefore we will in §3 introduce some concepts which act as a substitute for the topological index in the general case. The main results are established in §4.

2. Notation. The letters X and Y will always mean connected n-dimensional cohomology manifolds over the integers Z for some fixed n. For the definition, see [1, Chapter I]. If S is a locally compact Hausdorff space, $H^p(S)$ will be the p-dimensional cohomology group of S, in the sense of [2], with coefficients in Z and compact supports. We recall that $H^n(X)$ is isomorphic to Z or Z_2, according as X is orientable or not. If U is open in X, the standard homomorphism $H^p(U) \to H^p(X)$ is denoted by j^p_{XU}, or only by j, if there is no danger of misunderstanding. We recall that j^n_{XU} is surjective for every non-empty U. A domain in X is a connected open non-empty subset. The boundary $\bar{D} - D$ of a domain D is denoted by ∂D. The collection of all domains in X which have a compact closure is denoted by $J(X)$.

All mappings are assumed to be continuous. A mapping $f : X \to Y$ is acyclic if each point-inverse is compact and cohomologically trivial. It is monotone if each point-inverse is compact and connected. It is proper if every compact set in Y has a compact inverse-image. It is compact if it defines a proper mapping $X \to fX$. It is quasiopen if for each $y \in Y$ and for each open U containing a compact component of $f^{-1}(y)$, y is an interior point of fU with respect to Y.

3. Essential mappings. In the following definitions and lemmas we assume that a mapping $f : X \to Y$ is given.

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