A NOTE ON POLYHEDRA EMBEDDABLE IN THE PLANE

By Sibe Mardešić* and Jack Segal**

1. Introduction. In 1930 C. Kuratowski [9] characterized 1-dimensional polyhedra which are embeddable in the 2-sphere S^2 as those which do not contain either of the two primitive skew curves K_1 or K_2 . The polyhedron K_1 is the 1-skelton of a tetrahedron with midpoints of a pair of non-adjacent edges joined by a segment, and K_2 is the complete graph on five vertices. He also described the secondary skew curves K_3 and K_4 which are non-polyhedral 1-dimensional Peano continua (see §3). In 1937 S. Claytor [5] characterized Peano continua which are embeddable in S^2 as those which do not contain any one of the four skew curves K_i , $i = 1, 2, 3, 4$. Although Claytor's result applies to polyhedra, it makes use of K_3 and K_4 , and it seems desirable to have a polyhedral version which does not consider objects other than compact polyhedra. We do this by replacing K_3 and K_4 by a polyhedron $\perp$ called the "spiked disc". It consists of a disc and of an arc which have only one point in common and this point is an interior point of the disc and an end-point of the arc. We show in Theorem 1 that connected polyhedra which are embeddable in S^2 are characterized as those which do not contain any one of the three polyhedra K_1 , K_2 , $\perp$.

Along with embeddability the question of quasi embeddability has been considered. Given a compact metric space X, a space Y and a map $f : X \to Y$ into Y and given a number $\epsilon > 0$, we say that f is an ϵ-mapping of X into Y provided the diameter diam $f^{-1}(y) < \epsilon$, for each $y \in Y$. We say that X is quasi-embeddable in Y provided there is an ϵ-mapping $f : X \to Y$, for each $\epsilon > 0$. Quasi embeddability of compact metric spaces X is a topological property which does not depend on the given metric.

In 1933 S. Mazurkiewicz [10] proved that a 1-dimensional Peano continuum which does not contain a primitive skew curve is quasi embeddable in S^2. In particular, K_3 and K_4 are quasi embeddable in S^2 but not embeddable. Notice that on the other hand, embeddability always implies quasi embeddability. In a recent paper J. Segal [11, Theorem 3.6] has shown that for Peano continua which do not contain a secondary skew curve embeddability and quasi embeddability in S^2 are equivalent properties. The authors know of no characterization of Peano continua which can be quasi embedded in S^2. However, we are able to prove that connected polyhedra which are quasi-embeddable in S^2 are characterized as those which do not contain K_1 , K_2 or $\perp$.

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