CORRELATION OF SEMI-MULTIPLICATIVE FUNCTIONS

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1. Correlation functions. Let F and G be arithmetic functions, by which we understand complex-valued functions of a real variable which are zero if their arguments are not positive integers. For $n > 0$ it may be possible to define a new arithmetic function Ψ by

$$\Psi(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} F(k)G(k+n). \tag{1}$$

A statistician or an electrical engineer would call the function Ψ the *cross-correlation* of F and G (in that order). If $G = F$, Ψ is the *autocorrelation* of F. To make Ψ an arithmetic function we need to define $\Psi(n) = 0$ for $n \leq 0$, which departs from the usual definition of correlation function. However, we shall continue to use the above terminology for the function Ψ so defined.

In the definition of correlation function, the value $G(k + n)$ in (1) is sometimes replaced by its complex conjugate. In this case all results in this paper remain true when appropriately modified, with the exception of Theorem 4.

More general expressions of which (1) is a special case were studied by Mirsky [4]. From Mirsky's results we obtain an arithmetic formula for $\Psi(n)$ and a sufficient condition for its existence. Given F and G, define *associated* functions f and g by $F(n) = \sum_{d\mid n} df(d)$ and $G(n) = \sum_{d\mid n} dg(d)$. For given η, $0 \leq \eta < 1$, let $C(\eta)$ be the class of all arithmetic functions F such that the associated function f satisfies the two conditions

$$nf(n) = O(n^{\epsilon}) \qquad \text{for every positive } \epsilon \tag{2}$$

$$\sum_{n\leq x} n\,|f(n)| = O(x^{\eta}). \tag{3}$$

Using Theorem 2 in [4], we have

THEOREM 1. *If F and G both belong to some class $C(\eta)$, then $\Psi(n)$ exists for all $n \geq 1$, and*

$$\Psi(n) = \sum_{d\mid n} d\psi(d) \tag{4}$$

where

$$\psi(n) = \sum_{i} \sum_{\substack{j \\ (i,j)=n}} f(i)g(j). \tag{5}$$

The absolute convergence of $\sum_i f(i)$ and $\sum_j g(j)$ is a consequence of the fact that f and g satisfy (3). Indeed, one can show by partial summation that (3)

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