INFINITE MATRICES WHICH PRESERVE SCHAUDER BASES

BY WILLIAM RUCKLE

1. Introduction. Throughout this paper, X will represent a locally convex complete metric linear space (i.e. a space of type B_0 [1]) or a Banach space which has a Schauder basis, $\mathfrak{X} = (x_i)$. For our purposes the scalar field may be regarded as the real or complex numbers. Let $\mathfrak{F} = (f_i)$ be the sequence of coefficient functionals biorthogonal to $\mathfrak{X}$, $f_n(\sum_{i=1}^{\infty} a_i x_i) = a_n$; and let $A = [a_{ij}]$: $i, j = 1, 2, \cdots$ represent an infinite matrix with a_{ij} the element in the i-th row, j-th column. We assume $\sum_{i=1}^{\infty} a_i x_i$ converges in X to y_j for each j. Our object is to investigate conditions which are necessary and sufficient for $\mathfrak{Y} = (y_i)$ to be a basic sequence (a basis for cl sp $\mathfrak{Y}$, the closed linear span of $\mathfrak{Y}$) or a basis for X.

1.1. LEMMA. *If $\sum_{j=1}^{\infty} t_j y_j$ converges, $\sum_{j=1}^{\infty} a_{ij} t_j$ converges for each i, and $\sum_{j=1}^{\infty} t_j y_j = \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} a_{ij} t_j) x_i$.*

Proof. Each member, f_i, of $\mathfrak{F}$ is continuous by [5; 207, Theorem 1] so $f_i(\sum_{j=1}^{\infty} t_j y_j) = \sum_{j=1}^{\infty} t_j f_i(y_j) = \sum_{j=1}^{\infty} t_j a_{ij}$.

We shall denote by $||x||$ the distance from 0 to a given point x in X or the norm of x in the case of X, a Banach space.

1.2 THEOREM. *If X is a Banach space, $\mathfrak{Y}$ is basic if and only if there is a number $M > 0$ such that for every pair of positive integers $q > p$ and $t_1, t_2, \cdots, t_q$ arbitrary scalars*

$$\left|\left|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{p} t_j a_{ij}\right)x_i\right|\right| \le M \left|\left|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{q} t_j a_{ij}\right)x_i\right|\right|.$$

Proof. This follows from Lemma 1.1 and the known [5; 211, Theorem 5] fact that $\mathfrak{Y}$ is basic if and only if

$$\left|\left|\sum_{j=1}^{p} t_j y_j\right|\right| \le M \left|\left|\sum_{j=1}^{q} t_j y_j\right|\right|$$

where M, p, q and $t_1, \cdots, t_q$ are as in the statement of the theorem.

The conditions of Theorem 1.2 assure that $\mathfrak{Y}$ is a basis for X if and only if $\mathfrak{Y}$ is fundamental in X, i.e. cl sp $\mathfrak{Y} = X$.

2. Conditions on sequence spaces. By $s = (s_i)$ we mean the scalar sequence $\{s_1, s_2, \cdots\}$, which we shall treat as an infinite column vector. If S is a linear space of such sequences, let S' represent $\{t = (t_i)$: $\sum_{i=1}^{\infty} s_i t_i$ converges for each $s \in S\}$. Given a matrix $C = [c_{ij}]$ each row of which is in S', let $C(S) = \{t = Cs = (\sum_{j=1}^{\infty} c_{ij} s_j) : s \in S\}$.

Received September 10, 1965.