THE NON-EUCLIDEAN PYTHAGOREAN THEOREM WITH RESPECT TO THE BERGMAN METRIC

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1. Introduction. We consider the unit hypersphere **H** in the space $\mathbf{C}^n$ of n complex variables furnished with the Bergman metric $M(\mathbf{H})$. (Throughout this paper, n stands for any integer greater than or equal to 2.) In the space $(\mathbf{H}, M)$ there exists a unique shortest geodesic connecting two given distinct points in **H**. Therefore three points in $(\mathbf{H}, M)$ determine a non-euclidean triangle uniquely. A *non-euclidean triangle* consists of three sides given by the shortest geodesics and three angles given by the analytic angle (see §2 for definition). The length of a side is measured by $M(\mathbf{H})$.

The main purpose of the present paper is to establish certain differential geometric identities which hold on a non-euclidean right triangle in $(\mathbf{H}, M)$, one of which may be regarded as the non-euclidean Pythagorean theorem (see §4). These identities enable us to provide various interesting inequalities on non-euclidean triangles. Using the above considerations, we obtain distortion theorems for both the euclidean area and B-area of a triangle with the right analytic angle and for the euclidean lengths of a line segment and of a shortest geodesic line segment under certain biholomorphic interior mappings of **H**. These distortion theorems may be regarded as generalizations of the Schwarz lemma to the corresponding quantities.

2. Notation and preliminaries. We consider bounded domains **D** furnished with the Bergman metric

$$M(\mathbf{D}): \qquad ds^2_{\mathbf{D}}(z) = T_{\alpha\bar{\beta}}\, dz^\alpha\, d\bar{z}^\beta, \qquad z = (z^1, z^2, \cdots, z^n),$$

in the space $\mathbf{C}^n$ of n complex variables $z^1, z^2, \cdots, z^n$; here we use the summation convention. The components of the metric tensor of $M(\mathbf{D})$ have the form:

$$T_{\alpha\bar{\beta}} = \partial^2 \log K_{\mathbf{D}} / \partial z^\alpha\, \partial \bar{z}^\beta,$$

where $K_{\mathbf{D}} \equiv K_{\mathbf{D}}(z, \bar{z})$ is the Bergman kernel of the domain **D**. The metric $M(\mathbf{D})$ is positive definite and invariant under biholomorphic mappings of **D** (see [1], [5]).

Following B. Fuks [7], we define the *analytic angle* F between two vectors $u(u^\alpha)$ and $v(v^\alpha)$ in $(\mathbf{D}, M)$ by the equation

$$(2.1) \qquad \cos F = \frac{|[u, v]|}{||u||\ ||v||}, \qquad [u, v] = T_{\alpha\bar{\beta}} u^\alpha \bar{v}^\beta, \qquad ||u||^2 = [u, u],$$

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