TWO PROBLEMS OF HEWITT ON TOPOLOGICAL EXPANSIONS

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The purpose of this note is to answer two questions raised by Edwin Hewitt [1; 316, last line and 317, ll. 1-2 and ll. 8-10]. We shall produce a completely ordered family of completely normal spaces defined over the same basic set such that the least upper bound of the above family is not normal. This result, of course, answers both questions. For terminology, see [1].

Let $M = \{(x, y)\}$ be the set of all pairs of real numbers such that $y \geq 0$. We define a Niemytzki neighborhood basis for a point $(x, 0)$ as follows: a member of this basis consists of $(x, 0)$ and all points (x', y') with $(x' - x)^2 + (y' - r)^2 < r^2$ for some $r > 0$. The neighborhood basis is obtained by letting r take on all positive real values. Let ω_c be the smallest transfinite ordinal such that the set of ordinals preceeding ω_c has the cardinality of the set of all real numbers. We now use these ordinals to subscript all the real numbers: $x_0, x_1, \cdots, x_\omega, \cdots, x_\alpha, \cdots, \alpha < \omega_c$. Let M be endowed with that topology as a subspace of E_2 (this topology is thought of as having been defined by means of a neighborhood basis for each point). We call this topological space T_0. We now define T_ξ for $1 \leq \xi \leq \omega_c$ as follows: T_ξ has a neighborhood basis for each point which is the same as that obtained for T_0 except for each $(x_\lambda, 0)$, $0 \leq \lambda < \xi$, where one uses a Niemytzki neighborhood basis. Thus T_ξ is defined for all $0 \leq \xi \leq \omega_c$. Clearly, T_{ω_c} is the well-known Niemytzki space. T_{ω_c} is known to be not normal. We now prove the following.

LEMMA. *If T_ξ is completely normal, then $T_{\xi+1}$ is also completely normal.*

Proof. Let S be a subspace of $T_{\xi+1}$, we show that S is normal. Let A and B be disjoint closed sets in S. Now $T = S - \{(x_\xi, 0)\}$ is a subspace of both T_ξ and $T_{\xi+1}$. Thus $A \cap T$ and $B \cap T$ are disjoint closed sets in T. Since T (as a subspace of T_ξ) is normal, there exist sets U_A and U_B, open in T, such that $A \cap T \subset U_A$, $B \cap T \subset U_B$, and $U_A \cap U_B = \phi$. If $(x_\xi, 0) \notin S$, our proof is finished. If $(x_\xi, 0) \in S$, then without loss of generality, we may assume $(x_\xi, 0) \notin A$. Since A is closed, there exists V, a neighborhood of $(x_\xi, 0)$ in S, such that $V \cap A = \phi$. Since S is regular $\exists$ V_1 open in S which contains $(x_\xi, 0)$ such that $\bar{V}_1 \subset V$. (The "$^-$" represents closure in S). Thus $A \subset U_A - \bar{V}_1$, $B \subset U_B \cup V_1$ and $(U_A - \bar{V}_1) \cap (U_B \cup V_1) = \phi$. Since U_A and U_B are open in T and T is open in S, U_A and U_B are also open in S. Thus $U_A - \bar{V}_1$ and $U_B \cup V_1$ are open in S. Q.E.D.

We are now in a position to answer the problems proposed by Hewitt. Let $\mathfrak{A}$ be the class of all $T_\lambda (0 \leq \lambda \leq \omega_c)$ such that T_λ is not completely normal.

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