QUASICONFORMAL MAPPINGS WITH GIVEN BOUNDARY VALUES

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A continuous strictly increasing function μ mapping the real line onto itself is called ρ-quasisymmetric, $1 \leq \rho < \infty$ (this suggestive nomenclature was first employed by J. Kelingos) if

$$\frac{1}{\rho} \leq \frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)} \leq \rho \tag{1}$$

for all $t \neq 0$ and for all x. It was shown by Beurling and Ahlfors [1; 134-139]. (cf. [2] for a more detailed exposition of the computation) that any given ρ-quasisymmetric function μ has an extension to a K-quasiconformal self homeomorphism of the upper half-plane with

$$K \leq (\text{constant})\ \rho^2, \tag{2}$$

(The constant can be shown to be 1). The object of this paper is to improve the inequality (2) for large ρ.

Given the function $\mu = \mu(x)$, $-\infty < x < \infty$ let $K(\mu) = \inf\ \{K \mid$ there exists a K-quasiconformal extension of μ to the upper half-plane$\}$, and let $\Psi(\rho) = \sup\ \{K(\mu) \mid \mu$ is ρ-quasisymmetric$\}$. According to [2], $\Psi(\rho) = O(\rho^2)$ as $\rho \neq \infty$. Our main result is the following theorem.

THEOREM.

$$\Psi(\rho) = O(\rho), \qquad \rho \rightarrow \infty. \tag{3}$$

Remark. Relation (3) is best possible so far as order is concerned because as shown in [1; 132-134],

$$\Psi(\rho) \geq \frac{\log 2}{2}(\rho - 1),\ \rho > 4e + 1.$$

We use the formula

$$D + \frac{1}{D} = \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{u_x v_y - u_y v_x} \tag{4}$$

for the local dilatation $D = D(z)$ of any differentiable function $f = u + iv$ with positive Jacobian which maps one plane domain homeomorphically onto another. f is K-quasiconformal in a domain $\mathfrak{D}$ if $D(z) \leq K$ for all $z \in \mathfrak{D}$.

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