THE IMPLICIT FUNCTION THEOREM IN FUNCTIONAL ANALYSIS

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We consider a functional $f(y, z)$, or $f : Y_0 \times Z_0 \to F$, where F is a linear space, Y_0, Z_0 are subsets of linear spaces Y and Z of functions from an arbitrary space X into some Banach space E, and $Y \subset Z$, $Y_0 \subset Z_0$, (Y, Z Banach spaces, E finitely dimensional). The process usually associated with f may, or may not, involve a "loss of derivatives" in the terminology of J. Moser [6]. Under various sets of hypotheses we prove that there exists at least one function $\psi \in Y_0 \subset Z_0$ such that

$$f(\psi, \psi) = 0. \tag{1.1}$$

Under one of the sets of hypotheses taken into consideration, and for which the process usually associated with f involves no loss of derivatives, we prove a Theorem A (§2) one of whose corollaries (§5) includes a statement which we apply in a paper [1] on hyperbolic partial differential equations. A slight restriction of the hypotheses guarantees (Theorem B) a local uniqueness of the function ψ above. A number of applications are made. The proof of Theorem A is based on the remark that, by the usual argument of the implicit function theorem in functional analysis (see, for instance [5; 194–198]), an element $y_z \in Y_0$ can be proved to exist for every $z \in Z_0$, such that $f(y_z, z) = 0$, and that the ensuing map $\tau : Z_0 \to Y_0$ restricted to Y_0 can be proved to satisfy Schauder's fixed point theorem. The usual argument is here given for locally convex topological vector spaces.

In the case in which the process usually associated with f involves an actual loss of derivatives, the previous remark still applies and an analogous theorem can be proved (Theorem D) provided full use is made of the implicit function theorem that J. Moser [6] has recently proved—and applied to a number of problems. Also, use is made of other considerations of the papers of J. Nash and J. Schwartz [7] and [10] which were point of departure for the results of J. Moser [6]. Applications, other than those mentioned above, will follow. Considerations in the line of J. Leray's are also included (§2, No. 4).

§1. The Implicit Function Theorem of Functional Analysis

1. The equation $f(y) = 0$. Let Y be a Hausdorff, complete, locally convex topological vector space, and $f(y)$ a (not necessarily linear) functional $f : Y_0 \to F$ on a subset Y_0 of Y to a linear space F. We shall assume that an "approximate"

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