A NOTE ON UNITARY OPERATORS IN C^*-ALGEBRAS

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1. Introduction. We show that, in any C^*-algebra $\mathfrak{A}$, convex linear combinations of unitary operators are uniformly dense in the unit sphere of $\mathfrak{A}$. In other terms, the unit sphere in $\mathfrak{A}$ is the closed convex hull of its normal extreme points, even though non-normal extreme points will in general be present. This fact has several useful technical implications. For example, it follows that the norm of a linear mapping ϕ between C^*-algebras can be computed using only normal operators, that is, from the effect of ϕ on abelian $*$-subalgebras. In addition, we show that a linear mapping between C^*-algebras which conserves the identity and sends unitary operators into unitary operators is a C^*-homomorphism.

2. The main result. Let $\mathfrak{A}$ be a C^*-algebra, that is, a uniformly closed self-adjoint algebra of operators on some complex Hilbert space H. Throughout, we assume that $\mathfrak{A}$ contains the identity operator I. $U(\mathfrak{A})$ will denote the set of unitary operators in $\mathfrak{A}$, and $co(U(\mathfrak{A}))$ the convex hull of $U(\mathfrak{A})$.

Lemma 1. *In any von Neumann algebra M, co $(U(M))$ is weakly dense in the unit sphere of M.*

Proof. This follows readily from the known fact that, in a von Neumann algebra M with no finite summands, the weak closure of $U(M)$ is the unit sphere ([3, Theorem 1 et seq.]). For completeness, however, we include a proof of the lemma.

Let C denote the weak closure of $co(U(M))$. To show that C is the unit sphere, by Krein–Mil'man, it suffices to show that C contains all extreme points of the unit sphere. Using [5, Theorem 1], it follows readily that these are the partial isometries V in M such that, for some central projection D, $V^*V \geq D$ and $VV^* \geq I - D$. Therefore, replacing M by appropriate direct summands and noting that $C^* = C$, it suffices to consider the case $V^*V = I$. In addition, we can assume that $VV^* = P \neq I$. Given vectors x_i, y_i $(i = 1, \cdots, n)$ and $\epsilon > 0$, we will exhibit a unitary U in M such that $|((U - V)x_i, y_i)| < \epsilon$, for all i.

Let $\mathfrak{M}$ be the range of $I - P$. Then the $V^n\mathfrak{M}$ are mutually orthogonal $(n \geq 0)$ and the restriction of V to the orthogonal complement $\mathfrak{N}$ of $\bigoplus_{n=0}^{\infty} V^n\mathfrak{M}$ is unitary. Let Q_n be the projection on $V^n\mathfrak{M}$, and choose n such that $\|\sum_{k>n} Q_k x_i\| < \epsilon/2(1 + \max \|y_i\|)$, for all i. Let $U = V$ on the subspace $\mathfrak{N} \oplus \mathfrak{M} \oplus \cdots \oplus V^n\mathfrak{M}$, $= V^{*(n+1)}$ on $V^{n+1}\mathfrak{M}$, and $= I$ on $\bigoplus_{k>n+1} V^k\mathfrak{M}$. Then

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