A UNIQUENESS THEOREM FOR THE n-DIMENSIONAL HELMHOLTZ EQUATION

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1. Introduction. In this paper we prove the following

THEOREM. *Let* $u(x_1, x_2, \cdots, x_n)$ *be an everywhere twice continuously differentiable solution of the Helmholtz equation*

$$\sum_{k=1}^{n} \partial^2 u/\partial x_k^2 + u = 0 \qquad (n \geq 2) \tag{1.1}$$

and satisfy the integral condition

$$\lim_{R\to\infty} \int_\Omega \left| \int_0^R u(x) r^{(n-2)/2}\, dr \right| d\Omega = 0, \tag{1.2}$$

where Ω *is the surface of the n-dimensional unit sphere and* $x = (x_1, x_2, \cdots, x_n) = r\tau$ *(r being the spherical distance and* τ *the unit n-dimensional vector). Then* $u \equiv 0$ *throughout the entire* $(x_1, x_2, \cdots, x_n)$*-space.*

This theorem is a generalization to n dimensions of the analogous 2-dimensional theorem first proved by Owens in [6]. Related results especially with regard to existence have been established by Hartman and Wilcox in [2] and by Owens in [4], [5]. Furthermore, it is shown in [2] that any solution of (1.1) which satisfies the integral condition

$$\lim_{R\to\infty} \int_\Omega \left| \int_0^R u(x) r^{(n-2)/2}\, dr \right|^2 d\Omega = 0 \tag{1.3}$$

must be identically zero. It is easily seen using the inequality of Schwarz that the condition (1.3) implies (1.2) but not necessarily conversely. Hence, our theorem includes their result.

Our method of proof is similar to that used by Owens in [6]. A sketch of the proof will now be given. In order to show that u vanishes at a point P different from the origin one can proceed as follows. The Helmholtz equation and the integral condition remain invariant under orthogonal transformations. Hence, the ray joining the origin to the point P can be considered to be the x_1-axis. Now, since any twice continuously differentiable solution of the Helmholtz equation is an analytic function of all its arguments (indeed, solutions of elliptic type of any analytic nonlinear partial differential equation of the second order with n independent variables are analytic, [3]) u will be an analytic function

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