THE EXISTENCE AND STABILITY OF STATIONARY POINTS

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1. Let $y \varepsilon R^n$ and $f : R^n \to R^n$ a map of R^n into itself. This note concerns conditions sufficient to assure the existence of y_0 satisfying

$$f(y_0) = 0 \tag{1.1}$$

and the stability of the solution $y(t) \equiv y_0$ of the autonomous differential equation

$$y' = f(y). \tag{1.2}$$

The method will be a generalization and, at the same time, a simplification of that used to prove Theorem (Ia) in [1]. The procedure turns out to be a variant of Lyapunov's second method depending on a "Lyapunov vector function".

A result on (1.1), (1.2) obtained in §3 will be generalized in §5 to give sufficient conditions in order that a system

$$y' = f(t, y, z), \qquad z' = g(t, y, z)$$

have solutions $(y(t), z(t))$ [at least one or every] which exist for $t \geq 0$ and satisfy $z(t) \to 0$ at $t \to \infty$.

In the differential equation

$$y' = f(t, y), \tag{1.3}$$

let $f(t, y)$ be continuous on a (t, y)-region. If $V(t, y)$ is a real-valued function, its upper and lower trajectory derivatives relative to (1.3) at the point (t, y) are defined to be

$$V^*(t, y) = \limsup_{h \to +0} h^{-1}[V(t + h, y + h\, f(t, y)) - V(t, y)] \tag{1.4}$$

$$V_*(t, y) = \liminf_{h \to +0} h^{-1}[V(t + h, y + h\, f(t, y)) - V(t, y)]. \tag{1.5}$$

This note is motivated by the following considerations implicit in [1]. Consider the autonomous system (1.2) and the non-negative function

$$V(y) = |f(y)|. \tag{1.6}$$

If $f(y)$ is of class C^1 and $J(y) = (\partial f^i/\partial y^j)$ is its Jacobian matrix, then

$$V^*(y) = J(y)\, f(y) \cdot f(y)/|f(y)|$$

or $V^*(y) = 0$ according as $|f(y)| \geq 0$. Assume, as in [1], that for a suitable

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