TWO MONOTONIC, SINGULAR, UNIFORMLY ALMOST SMOOTH FUNCTIONS

By George Piranian

1. Introduction. A real-valued function f defined in some interval (a, b) is *smooth* at the point x in (a, b) provided

$$f(x + h) + f(x - h) - 2f(x) = o(h)$$

(see Rajchman [2; 23]). We shall say that f is *almost smooth* at x provided

$$f(x + h) + f(x - h) - 2f(x) = O(h),$$

and *uniformly almost smooth* in (a, b) $(-\infty \leq a < b \leq \infty)$ provided there exists a K such that

$$|f(x + h) + f(x - h) - 2f(x)| < Kh$$

for all x and h with $a < x - h < x + h < b$. Zygmund denoted the class of uniformly almost smooth functions by Λ^* [3; 47] and by Λ_* [4, Vol. 1; 43]. In [3] he investigated differentiability properties of the functions belonging to Λ^*, showed that the conjugate of a 2π-periodic function in Λ^* also belongs to Λ^*, and proved that uniform $O(1/n)$-approximation in $[0, 2\pi]$ to f by trigonometric polynomials T_n is possible if and only if $f \in \Lambda^*$.

The purpose of the present note is to furnish explicit descriptions of two singular monotonic functions in Λ^*, and to meet thus a need that arises in a paper by P. L. Duren, H. S. Shapiro, and A. L. Shields [1] on univalent holomorphic functions. (Added July 12, 1965: In §4, we show how the construction in §2 can be modified so that the resulting function is not merely almost smooth, but smooth.)

2. Piecewise cubic approximations.

Theorem 1. *There exists a strictly increasing, uniformly almost smooth function whose derivative, wherever it exists, is* 0 *or* ∞.

We shall construct the required function f in the interval $[0, 1]$ in such a way that it is uniformly almost smooth on $(-\infty, \infty)$ if it is extended by either of the two conditions

$$f(x + 1) = f(x) + 1$$

and

$$f(x) = \begin{cases} 0 & (x < 0), \\ 1 & (x > 1). \end{cases}$$

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