SOME CONCEPTS OF PARALLELISM WITH RESPECT TO A GIVEN TRANSFORMATION GROUP

By Charles Loewner

To Gabor Szegö on the occasion of his seventieth birthday.

1. Introduction. In the papers [1] and [2] the author introduced some transformation semigroups in the space of the hyperbolic geometry which may be considered as analogues of the translation group of the Euclidean geometry. Their introduction was based on semigroup-theoretical considerations. In this paper we take up the problem again but base our consideration on concepts of parallelism defined by postulates which are satisfied in case of the ordinary parallelism of the Euclidean geometry. We arrive at semigroups which comprise some of those considered in [1] and [2] as special cases.

The parallelism of the Euclidean geometry is a binary relation $\Pi(\alpha, \beta)$ in the set V of all vectors of the Euclidean space E_n which satisfies the following postulates and is characterized by them:

P_1^* : $\Pi(\alpha, \beta)$ is an equivalence relation in V.
P_2^* : The equivalence classes form continuous vector fields extended over the whole E_n .
P_3^* : The relation $\Pi(\alpha, \beta)$ is invariant under the group G of isometries of E_n .
P_4^* : If a and b are any two points of E_n , then to each vector α attached to a there exists a unique parallel vector β attached to b, and the mapping $\alpha \rightarrow \beta$ is linear.

There exists a unique parallelism in E_n satisfying the postulates P_1^* , P_2^* , P_3^* and P_4^* . If then, the vector fields representing equivalence classes are interpreted as infinitesimal transformations, the generated finite transformations form the ordinary translation group of E_n .

In the hyperbolic geometry there exists no parallelism satisfying the introduced postulates with the understanding that G is again the group of isometries. But one may ask, whether by weakening the postulates useful concepts can be obtained and construction of such possibilities for the hyperbolic and Euclidean geometries is the main contents of this paper. As a matter of fact, for any given Lie-group, G acting on a manifold M, we may ask whether there exists a concept of parallelism $\Pi(\alpha, \beta)$ satisfying at least the postulates:

P_1 : $\Pi(\alpha, \beta)$ is an equivalence relation in a nonempty set V of vectors. (We do not make the assumption that V is the totality of all vectors of M as is the case of the ordinary parallelism of the Euclidean geometry.)

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