PERIODS OF DIFFERENTIALS ON OPEN RIEMANN SURFACES

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The modern theory of square integrable differentials on open Riemann surfaces is due to L. Ahlfors [2]. This theory not only solved many problems but raised new ones as well. The present paper is concerned with a problem of the latter type. Can the various classes of differentials introduced by Ahlfors be characterized by their periods in some sense?

We obtain an affirmative answer to this question for several of the classes. As a by-product we obtain a simple proof of a very general Dirichlet principle originally due to M. Ohtsuka [6].

1. Preliminaries. Let c_i^n denote a singular n-simplex on a surface $W(n=0,1,2)$. A formal sum $c^n = \sum_{i=1}^{\infty} x_i c_i^n$, where each x_i is an integer, will be called a *relative n-chain* provided that for each compact set $K \subset W$ the set of indices i for which $x_i \neq 0$ and $|c_i^n| \cap K \neq \phi$ is finite; here $|c_i^n|$ is the image of c_i^n on W. The boundary homomorphism ∂ applied to a relative n-chain yields a relative $(n-1)$-chain. In this way relative cycles and boundaries can be defined. A relative 1-cycle is said to be weakly homologous to 0 if it is the boundary of a relative 2-chain.

If $c = \sum x_i c_i$ is a relative 1-chain and ω is a first order differential, we define $\int_c \omega = \sum x_i \int_c \omega$ provided each $\int_{c_i} |\omega| < \infty$ and the series converges absolutely. If $\rho\, |dz|$ is a linear density, we define $\int_c \rho\, |dz| = \sum |x_i| \int_{c_i} \rho\, |dz|$. In particular, if $\omega = a\, dx + b\, dy$, then $|\omega| = (|a|^2 + |b|^2)^{\frac{1}{2}}\, |dz|$ is a linear density and $|\int_c \omega| \leq \int_c |\omega|$.

We use the notion of extremal length [3] in the following form. If C is a family of relative 1-chains on a Riemann surface W and $\rho\, |dz|$ is a Borel measurable linear density, define

$$L(\rho, C) = \inf_{c \in C} \int_c \rho\, |dz|, \qquad A(\rho) = \iint_W \rho^2\, dx\, dy,$$

and the *extremal length* of C is

$$\lambda(C) = \sup_{\rho} \frac{L^2(\rho, C)}{A(\rho)},$$

where the supremum is taken over all with $\rho\, L(\rho, C) > 0$. A statement is said to hold for *almost all* $c \in C$ if the subfamily C' for which the statement is false has $\lambda(C') = \infty$ (see [5], [6]).

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